21.) $f(x) = \sec x$ $\qquad n = 2$

$f'(x) = \sec x \tan x$ $\qquad f(0) = 1$

$f''(x) = \sec x \sec^2 x + \tan^2 x \sec x$ $\qquad f'(0) = 0$

$f''(0) = 1$

$$P_2(x) = 1 + \frac{1(x-0)^2}{2!}$$

25.) $f(x) = \sqrt{x}$ $\quad n = 3 \quad c = 4$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\qquad f(4) = 2$

$f''(x) = \frac{-1}{4} x^{-3/2}$ $\qquad f'(4) = \frac{1}{4}$

$f'''(x) = \frac{3}{8} x^{-5/2}$ $\qquad f''(4) = \frac{-1}{4} \cdot 4^{-3/2} = \frac{-1}{32}$

$f'''(4) = \frac{3}{256}$

$$P_3(x) = 2 + \frac{1}{4}(x-4)^1 - \frac{1}{32}\frac{(x-4)^2}{2!} + \frac{3(x-4)^3}{256 \cdot 3!}$$

27.) $f(x) = \ln x,\ n = 4\ ,\ c = 2$

$f'(x) = \frac{1}{x}$

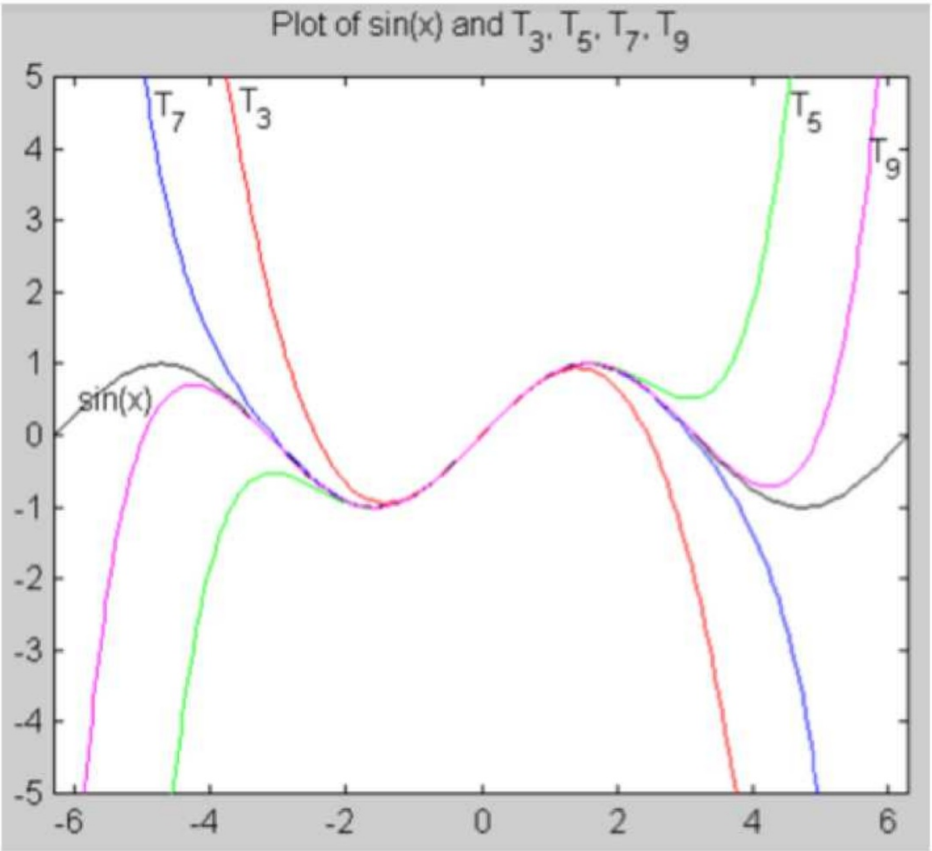
$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$c = 1$
$f(c) = 0$

$$P_4(x) = \ln 2 + \frac{1}{2}\frac{(x-2)^1}{} - \frac{1\,(x-2)^2}{4 \cdot 2} + \frac{1}{4}\frac{(x-2)^3}{\cdot 3}$$

$$f'''(x) = \qquad - \frac{3\,(x-2)^4}{8 \cdot 4}$$

Lagrange Error Bound



Error Bounds

*We use error bounds to determine how well or how poorly a Taylor Polynomial approximates a function.

Recall:

$$\boxed{\begin{aligned} \text{error} &= |S - S_n| = |f(x) - P_n(x)| \\ \text{error bound} &\geq |S - S_n| = |f(x) - P_n(x)| \end{aligned}}$$

$|f(x) - P_n(x)| \leq$ error bound

2 Types of Error Bounds for Taylor Polynomials

1. Alternating Series Remainder
2. Lagrange Error Bound

Review: Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_n \geq a_{n+1}$, then the absolute value of the remainder R_n involved in approximating the sum S by S_n is less than or equal to the first neglected term. That is,

first neglected term

$$\boxed{\text{Max Error} = |R_n| \leq |a_{n+1}|}$$

Lagrange Error Bound (a.k.a. Taylor's Theorem)

THEOREM 8.19 Taylor's Theorem

If a function f is differentiable through order $n + 1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$\text{Max Error} = |R_n| = \left| \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!} \right|$$

$$\text{Max Error} = \left|R_n\right| = \left|\frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}\right|$$

Where...

c = center

x = value used in the approximation

n = degree of the polynomial used

$f^{(n+1)}(z)$ = max of $f^{(n+1)}(x)$ on the interval [c, x] or [x, c]

Lagrange Error Bound WKST

max error

1. Let f be a function with 5 derivatives onthe interval [2, 3]. Assume $\left|f^{(5)}(x)\right| < 0.2$ for all x in the interval [2, 3] and that a fourth-degree Taylor polynomial for f at c = 2 is used to estimate $f(3)$.

f(3)

a) How accurate is this approximation? Round your answer to five decimal places.

0.2

$x = 3$

$c = 2$

$$\text{Max error} = \left|\frac{f^{(5)}(z)(3-2)^5}{5!}\right|$$

$$\left|\frac{.2 \cdot 1^5}{5!}\right| = \frac{1}{600}$$

.00167

*See printout.

1. Let f be a function with 5 derivatives onthe interval [2, 3]. Assume $\left|f^{(5)}(x)\right| < 0.2$ for all x in the interval [2, 3] and that a fourth-degree Taylor polynomial for f at c = 2 is used to estimate $f(3)$.

b) Suppose $\boxed{P_4(3) = 1.763}$. Use your answer from part (a) to find an interval in which must reside.

$$\left| f(3) - P_4(3) \right| \le .00167$$

$$\left| f(3) - 1.763 \right| \le .00167$$

$$-.00167 \le f(3) - 1.763 \le .00167$$

$$1.76133 \le f(3) \le 1.76467$$

1. Let f be a function with 5 derivatives onthe interval [2, 3]. Assume $\left|f^{(5)}(x)\right| < 0.2$ for all x in the interval [2, 3] and that a fourth-degree Taylor polynomial for f at c = 2 is used to estimate $f(3)$.

c) Could f(3) =1.778? Explain your reasoning.

No. 1.778 is not in the interval

By IVT...

1. Let f be a function with 5 derivatives onthe interval [2, 3]. Assume $\left|f^{(5)}(x)\right| < 0.2$ for all x in the interval [2, 3] and that a fourth-degree Taylor polynomial for f at c = 2 is used to estimate $f(3)$.

d) a.Could f(3)=1.764? Explain your reasoning.

Yes. 1.764 is in the interval

2. $f(x) = \sin x$

a) Find the fifth-degree Maclaurin polynomial for f(x).

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

2. $f(x) = \sin x$

b) Use the polynomial found in part (a) to approximate $\sin 1$.

$P_5(1) = 1 - \frac{1}{3!} + \frac{1}{5!} = .8417$

2. $f(x) = \sin x$ degree: 5

c) Use Taylor's Theorem to find the maximum error for your approximation.

$$\left| \frac{f^{(n+1)}(1)\, x^{n+1}}{(n+1)!} \right| = \left| \frac{f^{6}(1)\,(1)^{6}}{6!} \right|$$

$$= \left| \frac{1 \cdot 1^{6}}{6!} \right|$$

3. $f(x) = e^x$

a) Write the fourth-degree Maclaurin polynomial for f(x).

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

b) Using your answer from part (a), approximate the value of e.

e^1

$x = 1$

$$P_4(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$$

2.7083

3. $f(x) = e^x$

$n=4$

c) Find a Lagrange error bound for the maximum error involved in the approximation found in part (b).

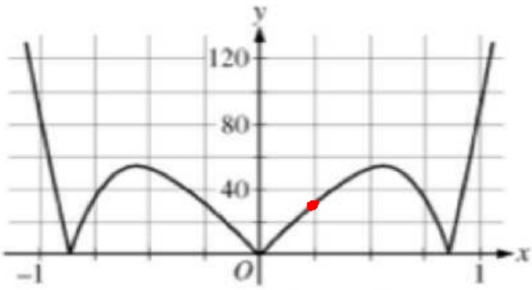
$$\left| \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!} \right| = \left| \frac{3 \cdot (1)^5}{5!} \right|$$

2.7083

$$\frac{1}{40}$$

5.

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

$f^{(5)}\left(\frac{1}{4}\right) = 40$

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$