# 9.7-9.8 Vector-Valued Functions

## What Are Vectors?

Quantities that have a magnitude and direction.

ex: <2, -4>

## What are Vector-Valued Functions?

A vector-valued function is a function that maps real numbers to vectors. By letting the parameter represent time, you can use a vector-valued function to represent motion along a curve.

ex: r(t)=<t, t-2>

## DEFINITION OF VECTOR-VALUED FUNCTION

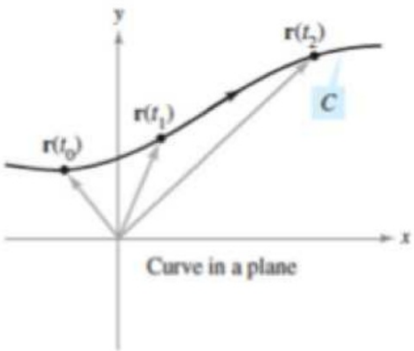
A function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \qquad \text{Plane}$$

or

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad \text{Space}$$

is a **vector-valued function,** where the **component functions** $f$, $g$, and $h$ are real-valued functions of the parameter $t$. Vector-valued functions are sometimes denoted as $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$.



Curve in a plane

AP Notation:

(f(t), g(t))

or

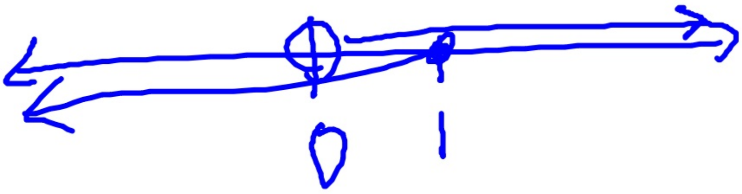
<f(t), g(t)>

## Domain

ex: Find the domain.

$$r(t) = \left\langle \ln t,\ \sqrt{1-t},\ \arctan t \right\rangle$$

$t > 0$ $\quad$ $t \leq 1$ $\quad$ $(-\infty, \infty)$

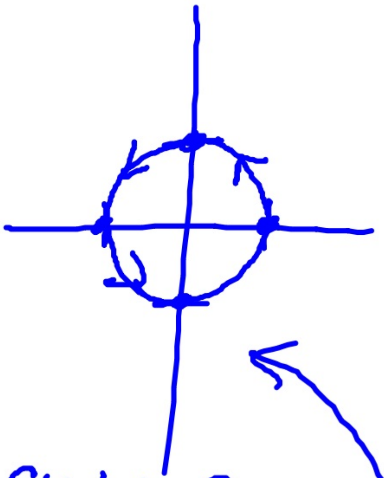



$(0, 1]$

## Evaluating

ex: Find the indicated values.

$$r(t) = (\cos t,\ \sin t)$$



$r(0) =$ (1, 0)

$r\left(\frac{\pi}{2}\right) =$ (0, 1)

$r(\pi) =$ (−1, 0)

$r\left(\frac{3\pi}{2}\right) =$ (0, −1)

Sketch for the plane curve

Sketching

ex: Sketch the plane curve.

$$r(t) = (\cos t,\ \sin t)$$

## Limits

ex: Find the indicated limit.

$$r(t) = \left\langle 5\arctan t,\ \frac{3t^2 - 7}{\sqrt{36t^4 - 16}},\ \left(1 + \frac{1}{t}\right)^{2t} \right\rangle$$

$$\lim_{t \to \infty} r(t) = \left\langle \frac{\pi}{2}, \frac{1}{2}, e^2 \right\rangle$$

Derivatives

$$\frac{-u'}{\sqrt{1-u^2}}$$

$y = a^u$

$y' = \ln a \cdot a^u \cdot u'$

ex: Find the indicated derivative.

$$r(t) = \left\langle \arccos 2t,\ 3^t,\ t^3 \right\rangle$$

a) $r'(t) = \left\langle \frac{-2}{\sqrt{1-4t^2}},\ \ln 3 \cdot 3^t,\ 3t^2 \right\rangle$

$-2\left(1-4t^2\right)^{-1/2}$

$1\left(1-4t^2\right)^{-3/2} \cdot -8t$

b) $r''(t) = \left\langle \frac{-8t}{\left(1-4t^2\right)^{3/2}},\ (\ln 3)^2 \cdot 3^t,\ 6t \right\rangle$

## WARNING:

While parametric equations and vectors are similar, finding the process for finding a second derivative of a vector and the second derivative of a parametric equation is different!!!

| Parametric Equations | Vectors |
|---|---|
| $x = f(t) \quad y = g(t)$ | $r(t) = \langle f(t), g(t) \rangle$ |
| $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$ | $r''(t) = \langle f''(t), g''(t) \rangle$ |

Integrals

ex: If $r'(t) = \left\langle \cos 2t,\ -2\sin t,\ \dfrac{3}{1+t^2} \right\rangle$ and $r(0) = \langle 5,\ 0,\ -1 \rangle$, find $r(t)$.

$r(t) = \left\langle \frac{1}{2}\sin 2t + 5,\ 2\cos t - 2,\ 3\arctan t - 1 \right\rangle$

PVAJ Vectors

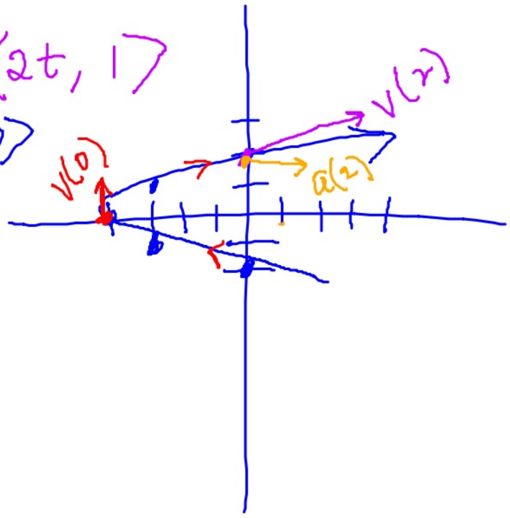
Position: $x(t) = s(t) = \langle f(t),\ g(t) \rangle$

Velocity: $v(t) =$ $\langle f'(t), g'(t) \rangle$

Acceleration: $a(t) =$ $\langle f''(t), g''(t) \rangle$

ex: $x(t) = \langle t^2 - 4, t \rangle$ $x'(t) = \langle 2t, 1 \rangle$

$x''(t) = \langle 2, 0 \rangle$




a) Sketch.

$x(-2) = \langle 0, -2 \rangle$

$x(-1) = \langle -3, -1 \rangle$

$x(0) = \langle -4, 0 \rangle$

$x(1) = \langle -3, 1 \rangle$

$x(2) = \langle 0, 2 \rangle$

b) Sketch $v(0) = \langle 0, 1 \rangle$

c) Sketch $v(2) = \langle 4, 1 \rangle$

d) Sketch $a(2) = \langle 2, 0 \rangle$

Rest

$$v(t) = 0$$

$$v(t) = \langle f'(t),\ g'(t) \rangle$$

A particle is at rest if and only if:

$$f'(t) = 0 \ \ \& \ \ g'(t) = 0$$

AT THE SAME TIME.

ex: $v(t) = \left\langle t - t^3,\ \dfrac{2t}{1+t^2} \right\rangle$

At what time(s) is the particle at rest?

$\boxed{t=0}$

$t - t^3 = 0$

$t(1-t^2) = 0$

$\boxed{t = 0,}\ \pm 1$

$\dfrac{2t}{1+t^2} = 0$

$\boxed{t = 0}$

in common

ex: $\dfrac{dx}{dt} = \sqrt{t^2 + 9}$ $\qquad$ $\dfrac{dy}{dt} = 2e^t + 5e^{-t}$

a) Find the velocity vector at t=0.

$\langle 3, 7 \rangle$

$-3.\underline{000}\cancel{0007}$

b) Find the acceleration vector at t=0.

$\langle 0, -3 \rangle$

ex: $\frac{dx}{dt} = \sqrt{t^2+9}$ $\qquad$ $\frac{dy}{dt} = 2e^t + 5e^{-t}$

x-coord.

c) At t=2 the particle is at (9,3). Find the x-coordinate at t=0.

$$x(0) = 9 + \int_2^0 \sqrt{t^2+9}\,dt = 2.581$$

d) At t=2 the particle is at (9,3). Find the y-coordinate at t=4.

$$y(4) = 3 + \int_2^4 y'(t)\,dt = 98.003$$

ex: $\dfrac{dx}{dt} = \sqrt{t^2+9}$ $\qquad$ $\dfrac{dy}{dt} = 2e^{t} + 5e^{-t}$

e) Find the slope at t=0.

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{7}{3} = \left.\frac{dy}{dx}\right|_{t=0}$$

$$y(0) = 3 + \int_{2}^{0} y'(t)\,dt$$

$$= -14.101$$

f) Write the equation of the tangent line at t=0 if at t=2 the particle is at (9,3).

$m = 7/3$

$(2.581, \quad )$

$$y + 14.101 = \frac{7}{3}(x - 2.581)$$

ex: $\dfrac{dx}{dt} = \sqrt{t^2+9}$ $\qquad$ $\dfrac{dy}{dt} = 2e^t + 5e^{-t}$

g) Discuss the concavity at t=1. Justify your answer.

$$\frac{dy}{dx} = \frac{2e^t + 5e^{-t}}{\sqrt{t^2+9}} = Q$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}(Q)}{\sqrt{t^2+9}}$$

$Y_1$

.287

ccu at $t=1$ since $.287 > 0$

Speed (Magnitude)

$$\boxed{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

Arclength (Distance)

$$\boxed{\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

 ex: $x(t) = \langle 2\sin t,\ \cos t \rangle$

a) Find the speed at $t = \pi$.

$$\sqrt{(2\cos\pi)^2 + (-\sin\pi)^2} = 2$$

b) Find the arc length on the interval $[0, \pi]$.

$$\int_0^{\pi} \sqrt{4\cos^2 t + (-\sin t)^2}\, dt = 4.844$$

ex:

At time $t \geq 0$, a particle moving in the $xy$-plane has velocity vector given by $\boxed{v(t) = \left\langle 4e^{-t}, \sin\left(1+\sqrt{t}\right)\right\rangle}$. What is the total distance the particle travels between $t = 1$ and $t = 3$?

(A) 1.861

(B) 1.983

(C) 2.236

(D) 4.851

$$\int_1^3 \sqrt{\left(4e^{-t}\right)^2 + \left(\sin(1+\sqrt{t})\right)^2}\, dt$$