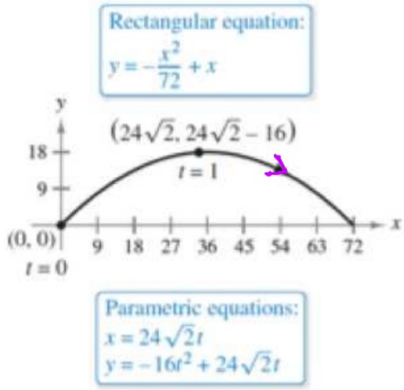
9.2-9.3 Parametric Equations

Parametric Equations

$$\boxed{\begin{array}{ccc} x = f(t) & & x = f(\theta) \\ & \text{or} & \\ y = g(t) & & y = g(\theta) \end{array}}$$

* t and θ are called the <u>parameter</u>

*the graph is called a <u>plane curve</u>



Think of a curve being traced out over time, somestimes doubling back on itself or crossing itself. Such a curve cannot be described by a function $y = f(x)$. Instead, we will describe our position along the curve at time t by

$x = x(t)$
$y = y(t)$

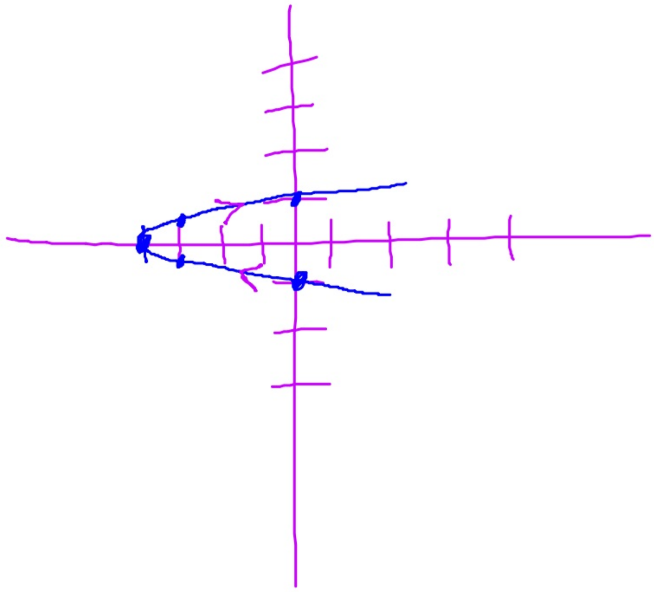
Then x and y are related to each other through their dependence on the parameter t.

ex: Sketch.

a) $x = t^2 - 4$

$y = \frac{t}{2}$

t	x	y
-2	0	-1
-1	-3	$-1/2$
0	-4	0
1	-3	$1/2$
2	0	1

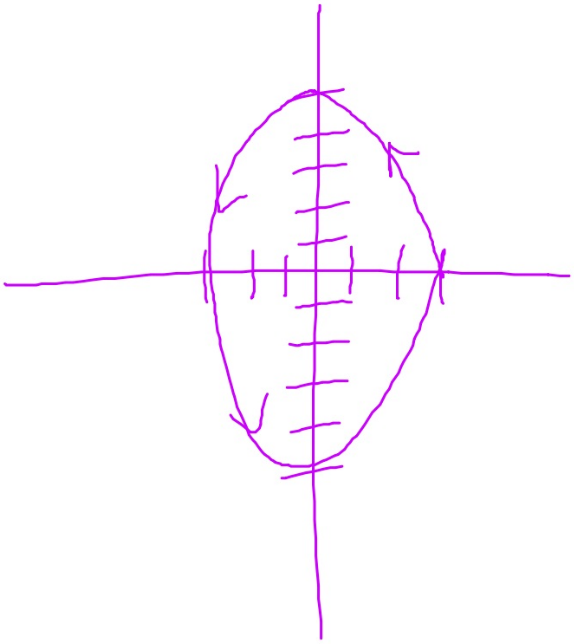


ex: Sketch.

b) $x = 3\cos\theta$

$y = 5\sin\theta$

θ	x	y
0	3	0
$\pi/2$	0	5
π	-3	0
$3\pi/2$	0	-5
2π		



Review

Horizontal Asymptotes: $y = a$

$$\lim_{x \to -\infty} f(x) = a \quad \text{or} \quad \lim_{x \to \infty} f(x) = a$$

Vertical Asymptotes: $x = b$

$$\lim_{x \; b^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to b^+} f(x) = \pm\infty$$

Finding Horizontal Asymptotes

- Isolate the parameter in the x equation.
- Find the limits:

$$\lim_{x\to -\infty} f(x) = l \quad \text{or} \quad \lim_{x\to\infty} f(x) = m$$

- Plug l and m into the y equation.

Finding Vertical Asymptotes

- Isolate the parameter in the y equation.
- Find the limits:

$$\lim_{y \to -\infty} f(x) = n \quad \text{or} \quad \lim_{y \to \infty} f(x) = p$$

- Plug n and p into the x equation.

ex: Find the asymptotes, then sketch.

a) $x = 1 + \frac{1}{t}$

$t = \cancel{y} + 1$ $y = t - 1$

$X = 1 + \frac{1}{y+1}$

$X - 1 = \frac{1}{y+1}$

$\frac{1}{X-1} = y + 1$

$\frac{1}{X-1} - 1 = y$

HA $\boxed{y = -1}$

$X = 1 + \frac{1}{t}$

$\frac{1}{X-1} = t$

$\lim_{X \to -\infty} \frac{1}{X-1} = 0$

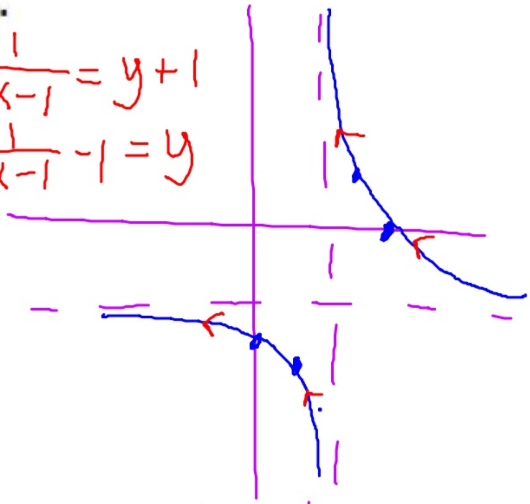
$\lim_{X \to \infty} \frac{1}{X-1} = 0$

VA $\boxed{X = 1}$

$t = y + 1$

$\lim_{y \to -\infty} (y+1) = -\infty$

$\lim_{y \to \infty} (y+1) = \infty$



t	x	y
-2	1/2	-3
-1	0	-2
1	2	0
2	3/2	1

ex: Find the asymptotes, then sketch.

b) $x = \ln 2t$

$y = \tan^{-1}(2t)$

HA

$e^{x} = e^{\ln 2t}$

$e^{x} = 2t$

$\frac{e^x}{2} = t$

$\lim_{x\to-\infty} \frac{e^x}{2} = 0$

$\lim_{x\to\infty} \frac{e^x}{2} = \infty$

$\boxed{\begin{array}{l} y=0 \\ y=\frac{\pi}{2} \end{array}}$

$y = \tan^{-1}(2\cdot 0) = 0$

$y = \tan^{-1}(2\cdot\infty) = \frac{\pi}{2}$

VA

$y = \tan^{-1}(2t)$

$\tan y = \tan(\tan^{-1}(2t))$

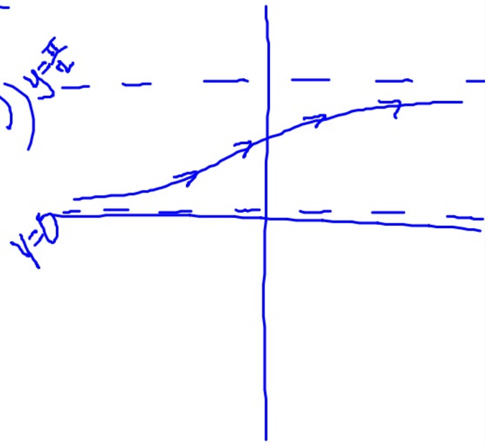
$\tan y = 2t$

$\frac{\tan y}{2} = t$

$\lim_{x\to-\infty} \frac{\tan y}{2} = N/A$

$\lim_{x\to\infty} \frac{\tan y}{2} = N/A$

$\boxed{\text{No VA}}$



ex: Find the asymptotes.

$$x = t + 5$$

$$y = \frac{6t}{4t - t^3}$$

Domain and Range

$$x = f(t)$$
$$y = g(t)$$

- Domain: the range of the x equation
- Range: the range of the y equation

ex: Find the domain and range.

a) $x = t^2 - 4$ D: $[-4, \infty)$ $\cup$ $t \in \mathbb{R}$

$y = \frac{t}{2}$ R: $(-\infty, \infty)$

b) $x = 2\cos\theta$ D: $[-2, 2]$ $\theta \in \mathbb{R}$

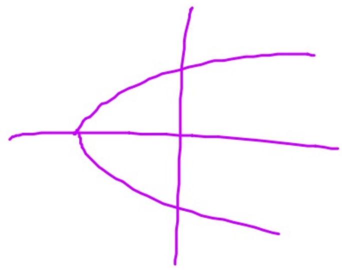
$y = -7\sin\theta$ R: $[-7, 7]$

c) $x = \sqrt{t}$ D: $[0, \infty)$ $t \geq 0$

$y = t + 2$ R: $[2, \infty)$

Eliminating the Parameter - Methods

1. Substitution
2. Trigonometric Identities



ex: Eliminate the parameter.

a) $x = t^2 - 4$

$y = \frac{t}{2}$

$t = 2y$

$x = (2y)^2 - 4$

$x + 4 = 4y^2$

$y = \pm \frac{\sqrt{x+4}}{2}$

ex: Eliminate the parameter.

b) $x = 3\cos\theta$

$y = 5\sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{x}{3} = \cos\theta \quad \frac{y}{5} = \sin\theta$

$\frac{y^2}{25} + \frac{x^2}{9} = 1$

ex: Eliminate the parameter.

c) $x = \tan\theta$

$y = \sec^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

$1 + x^2 = y$

ex: Eliminate the parameter.

d) $x = 2e^t$

$y = \dfrac{3}{e^t}$

$e^t = \frac{3}{y}$

$x = 2\left(\frac{3}{y}\right)$

$x = \frac{6}{y}$

or $y = \frac{6}{x}$ or $xy = 6$

Parametric Equations and Calculus

$$x = f(t)$$

$$y = g(t)$$

<u>1st Derivative:</u> $\dfrac{dy}{dx} = ?$

$$\frac{d}{dx}\left(g(t)\right) = g'(t) \cdot \frac{dt}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

2nd Derivative: $\dfrac{d^2y}{dx^2} = ?$ $\dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] \cdot \dfrac{dt}{dx}$

$$= \frac{\frac{d}{dx}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

ex: Find $\frac{dy}{dx}$ without eliminating the parameter.

$$x = \sqrt{t} \qquad x' = \frac{1}{2\sqrt{t}}$$

$$y = \sqrt{t-1} \qquad y' = \frac{1}{2\sqrt{t-1}}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{t-1}}}{\frac{1}{2\sqrt{t}}} = \frac{\sqrt{t}}{\sqrt{t-1}}$$

ex: Find $\frac{d^2y}{dx^2}$ without eliminating the parameter.

$$\frac{dy}{dx} = t^{3/2}$$

$$x = \sqrt{t} \qquad x' = \frac{1}{2\sqrt{t}}$$

$$y = \frac{1}{4}\left(t^2 - 4\right) \qquad y' = \frac{t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2}t^{1/2}}{\frac{1}{2\sqrt{t}}} = 3t$$

ex: Find the slope at $t = \dfrac{\pi}{3}$.

$$x = 2\sin t$$

$$y = 5\cos t$$

$$\frac{dy}{dx} = \frac{-5\sin t}{2\cos t}$$

$$-\frac{5}{2}\tan t$$

$$\boxed{\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{-5\sqrt{3}}{2}}$$

ex: Find the slope at the point (3,0).

When you find t, the solutions have to be solutions for both x and y

$$\frac{dy}{dx} = \frac{20t^3 - 5}{2t}$$

$$x = t^2 + 2$$

$$y = 5t^4 - 5t$$

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{15}{2}$$

(3, 0)

$x = t^2 + 2$	$0 = 5t(t^3 - 1)$
$3 = t^2 + 2$	$t = 0, 1$
$1 = t^2$	
$\pm 1 = t$	

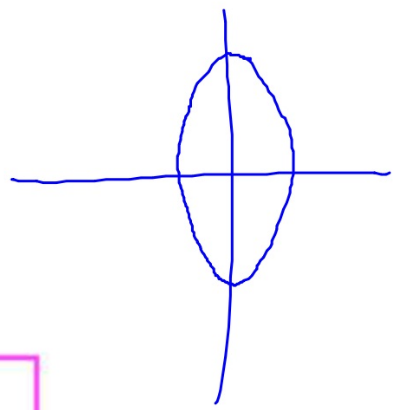
$\boxed{t = 1}$

ex: Write an equation of the tangent line at $t = \dfrac{\pi}{3}$.

$$x = 2\sin t \qquad \left(\sqrt{3}, \frac{5}{2}\right)$$

$$y = 5\cos t$$

Horizontal and Vertical Tangents



- Horizontal Tangent: $\frac{dy}{dx} = 0$

$$\boxed{\frac{dy}{dt} = 0 \text{ but } \frac{dx}{dt} \neq 0}$$

- Vertical Tangent: $\frac{dy}{dx}$ is undefined

$$\boxed{\frac{dx}{dt} = 0 \text{ but } \frac{dy}{dt} \neq 0}$$

ex: Find the times, t, at which the plane curve has a horitonal or vertical tangent.

$$x = \frac{t^3}{3} - t + 4$$

$$y = \frac{t^3}{3} + \frac{t^2}{2} - 2t + 1$$

$$\frac{dy}{dx} = \frac{t^2 + t - 2}{t^2 - 1}$$

$$= \frac{(t+2)(t-1)}{(t+1)(t-1)}$$

HT: $t = -2, \cancel{1}$

VT: $t = -1, \cancel{1}$

Concavity

- CCU: $\dfrac{d^2y}{dx^2} > 0$

- CCD: $\dfrac{d^2y}{dx^2} < 0$

ex: For what times, t, is the plane curve concave up and concave down.

$$x = \sqrt{t}$$

$$y = \frac{1}{4}\left(t^2 - 4\right)$$

$t \geq 0$

$$\frac{d^2y}{dx^2} = 3t$$

$+$

0

ccu

Arc Length

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

ex: Find the length of the curve on the interval [0, 2].

$x = t^2$

$y = t^3$

$$S = \int_0^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt$$

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