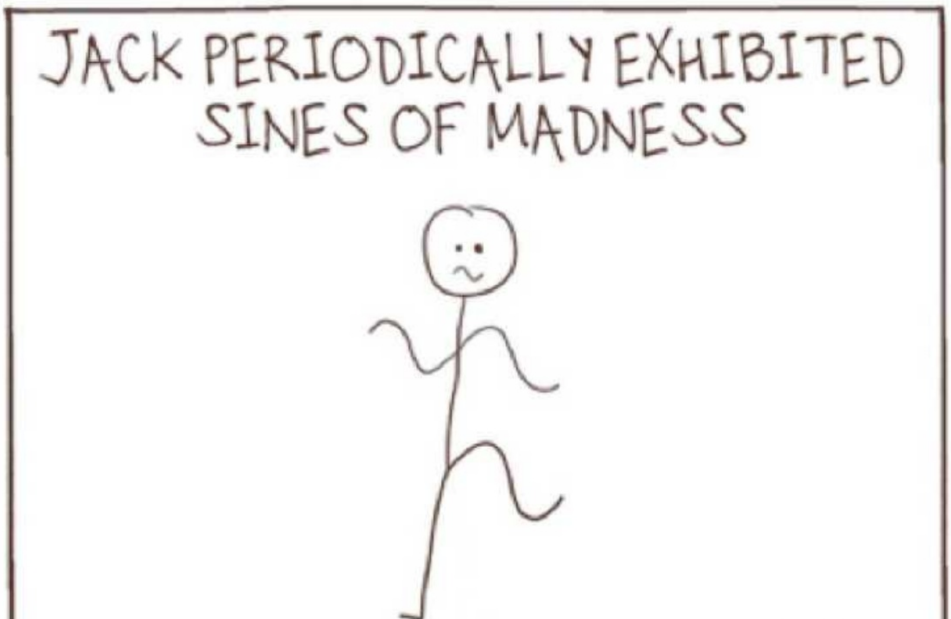
8.9 Representation of Functions by Power Series



Geometric Power Series

Review:

Geometric Series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1$

ex: Find a power series for f(x) with the given center.

a) $f(x) = \frac{1}{1-x}, \quad c = 0$ $\quad |x| < 1$

$$\sum_{n=0}^{\infty} x^n$$

ex: Find a power series for f(x) with the given center.

b) $f(x)=\dfrac{1}{1-x}, \quad c=8$

Steps:

1) Consider center

2) Denominator must have a subtraction

3) 1st term in denominator must be 1

$$\frac{1}{-7-(x-8)}$$

$$\frac{\frac{1}{-7}}{\frac{-7}{-7}-\left(\frac{x-8}{-7}\right)}$$

$$\frac{-\frac{1}{7}}{1-\left(\frac{x-8}{-7}\right)} = -\frac{1}{7}\left(\frac{1}{1-\left(\frac{x-8}{-7}\right)}\right)$$

$$\sum_{n=0}^{\infty} -\frac{1}{7}\left(\frac{x-8}{-7}\right)^n$$

$$\sum_{n=0}^{\infty} -\frac{1}{7}\left(\frac{8-x}{7}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{-1}{7}\cdot(-1)^n\left(\frac{x-8}{7}\right)^n$$

ex: Find a power series for f(x) with the given center.

c) $f(x) = \dfrac{4}{7+x}, \quad c = -6$

$$|x+6| < 1$$

$$-1 < x+6 < 1$$

$$-7 < x < -5$$

$$\frac{4}{1-(-x-6)}$$

$$\sum_{n=0}^{\infty} 4(-x-6)^n = \sum_{n=0}^{\infty} 4\left(-(x+6)\right)^n$$

$$= \sum_{n=0}^{\infty} 4(-1)^n (x+6)^n$$

ex: Find a power series for f(x) with the given center.

d) $f(x)=\dfrac{6}{4+x}, \quad c=2$

$$=\frac{\frac{6}{6}}{\frac{6}{6}-\left(\frac{-(x-2)}{6}\right)}$$

$$=\frac{1}{1-\left(\frac{-(x-2)}{6}\right)}$$

$$\sum_{n=0}^{\infty}\left(\frac{-(x-2)}{6}\right)^{n}$$

ex: Find a power series for f(x) with the given center.

e) $f(x)=\dfrac{4}{1-5x}, \quad c=3$

$$= \frac{4 / -14}{\frac{-14}{-14} - 5\left(\frac{x-3}{-14}\right)}$$

$$= \frac{-\frac{2}{7}}{1-5\left(\frac{x-3}{-14}\right)}$$

$$\sum_{n=0}^{\infty} -\frac{2}{7}\left(\frac{5(x-3)}{-14}\right)^n$$

Interval of Convergence and Geometric Power Series

We know...

Geometric Series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1$

To find the IOC of a Geometric Series solve: $|r| < 1$

*You don't need to use ratio test!

*Endpoints ALWAYS DIVERGE!

ex: Find the IOC.

a) $\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{5}\right)^n$

$$\left|\frac{x+1}{5}\right| < 1$$

$$-5 < x+1 < 5$$

$$-6 < x < 4$$

ex: Find the IOC.

b) $\sum_{n=0}^{\infty} 5x^n$

$|x| < 1$

$-1 < x < 1$

Operations with Power Series

Operations with Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$ $(Kx)^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

ex: Find a power series for f(x) with the given center. Then state the IOC.

a) $f(x)=\dfrac{3x-1}{x^2-1}, \quad c=0$

$$\frac{a_1}{1-r}$$

$$f(x) = \frac{3x-1}{(x+1)(x-1)} = \frac{\overset{2}{A}}{x+1} + \frac{\overset{1}{B}}{x-1}$$

$$\boxed{\begin{array}{c}\text{IOC} \\ (-1, 1)\end{array}}$$

$$f(x) = \frac{2}{x+1} + \frac{1}{x-1}$$

$$= \frac{2}{1-(-x)} + \frac{-1}{1-x}$$

$$f(x) = \sum_{n=0}^{\infty}\left(2(-x)^n + -1x^n\right) = \sum_{n=0}^{\infty}\left(2(-1)^n x^n - x^n\right)$$

$$= \sum_{n=0}^{\infty}\left[x^n\left(2(-1)^n - 1\right)\right]$$

ex: Find a power series for f(x) with the given center. Then state the IOC.

IOC
(0, 2)

$$\frac{a_1}{1-r}$$

b) $f(x) = \ln x, \quad c = 1$

$$f'(x) = \frac{1}{x} = \frac{1}{1-(-x+1)} = \frac{1}{1-(-(x-1))}$$

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$f(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

$$\ln x = C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

$x = 1;$
$C = 0$

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

IOC

$(0, 2]$

Check

$x = 0$	$x = 2$
$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n+1}}{n+1}$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$
$\sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1}$	conv. by AST
div. (LCT)	

ex: Find a power series for f(x) with the given center. Then state the IOC.

c) $f(x) = \dfrac{5}{4x^2+1}, \quad c = 0$

$$f(x) = \frac{5}{1-(-4x^2)}$$

$$f(x) = \sum_{n=0}^{\infty} 5(-4x^2)^n$$

$$|4x^2| < 1$$

$$x^2 < \frac{1}{4}$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

ex: Find a power series for f(x) with the given center. Then state the IOC.

d) $f(x)=\dfrac{1}{(1-x)^2}, \quad c=0$

$\int (1-x)^{-2}\,dx$

$\dfrac{1}{1-x}+C$

$\int f(x)\,dx = C + \sum_{n=0}^{\infty} x^n$

$f(x) = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$

$x=-1$	$x=1$
div.	div.

ex: Find a power series for f(x) with the given center. Then state the IOC.

e) $f(x) = \tan^{-1} x, \quad c = 0$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$f(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

IOC: $\boxed{[-1, 1]}$

FR 16

Let f be the function defined by $f(x) = \frac{1}{x-1}$.

(a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

(b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.