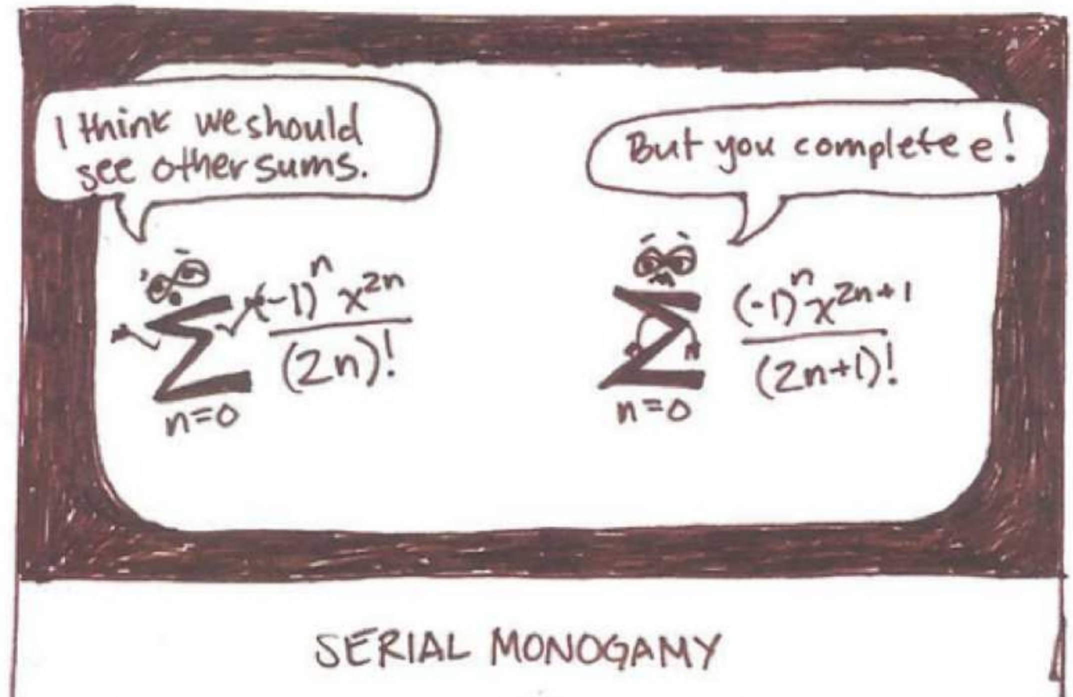
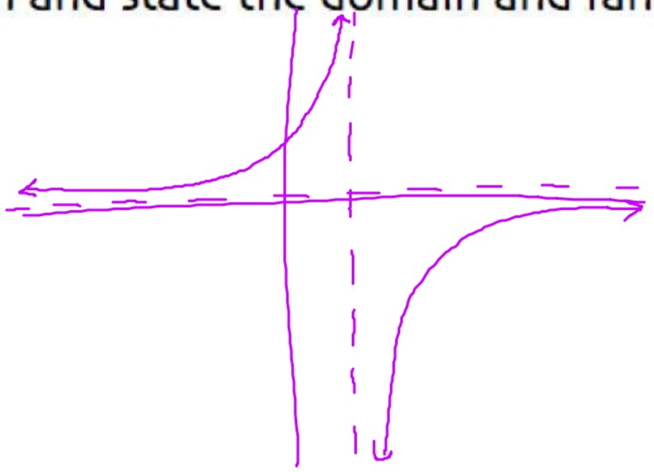
8.8 Power Series



ex: $f(x) = \dfrac{1}{1-x}$

a) Sketch and state the domain and range.



D: $\{x \mid x \neq 1\}$

R: $\{y \mid y \neq 0\}$

ex: $f(x) = \dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots$

geometric $r = x$

$|x| < 1$

$\boxed{-1 < x < 1}$

interval of convergence

b) Rewrite using long division.

$$
\begin{array}{r|l}
 & 1 + x + x^2 + \ldots\ldots \\
\hline
1 - x & 1 \\
 & \underline{-1 + x} \\
 & \quad x \\
 & \quad \underline{-x + x^2} \\
 & \qquad x^2 \\
 & \qquad \underline{-x^2 + x^3} \\
 & \qquad\qquad x^3
\end{array}
$$

 ex: $f\left(x\right)=\dfrac{1}{1-x}$

c) Using your calculator graph f(x), S_1, S_2, S_3, etc. On what interval does S=f(x)?

Power Series

A power series is a polynomial with infinitely many terms.

Definition of Power Series

centered at 0

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series.** More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at c,** where c is a constant.

ex: Identify the center.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n (2n-1) x^n}{2^n n!}$ $(x-0)^n$ $C=0$

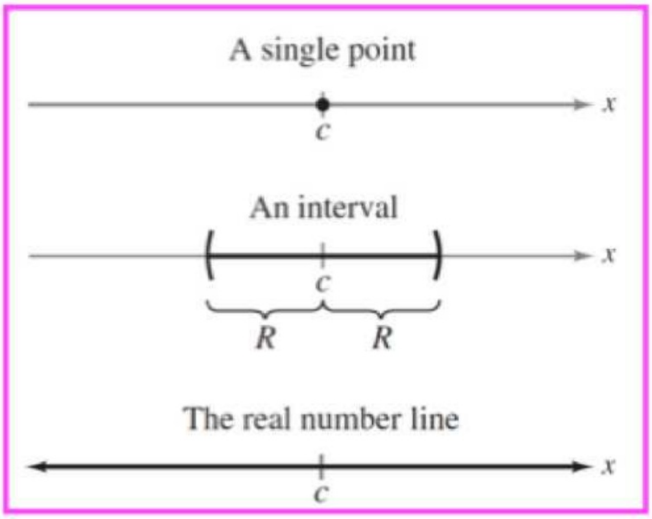
b) $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n^3}$ $C=3$

c) $(x+1)+\frac{(x+1)^2}{2}+\frac{(x+1)^3}{3}+\frac{(x+1)^4}{4}+\ldots$ $C=-1$

Interval of Convergence (IOC)

The IOC is a set of x-values in which a power series converges.

The IOC comes in three basic forms...



Finding The Interval of Convergence

To find the interval of convergence use the RATIO TEST.

ex: Identify the center and find the IOC.

n: variable
x: constant

a) $$\sum_{n=0}^{\infty} n!x^n = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$= \lim_{n\to\infty}|(n+1)x| = \infty > 1 \text{ diverges}$$

(if x=0 ?)

only converge at 0

C=0

ex: Identify the center and find the IOC.

$C = 0$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

$$\lim_{n\to\infty} \left| \frac{x^{n+1} \cdot n}{(n+1) \cdot x^n} \right| = \lim_{n\to\infty} \left| \frac{x \cdot n}{n+1} \right| = |x| < 1$$

$-1 < x < 1$

Check endpoints

$\boxed{[-1, 1)}$

$x = -1$

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (A.S.T.)

$x = 1$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverge (harmonic)

ex: Identify the center and find the IOC.

$\to c = 0$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{\underbrace{\cancel{(2(n+1)+1)}!}_{(2n+3)!}} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0$$

$0 < 1$

converge

IOC: $(-\infty, \infty)$

ex: Identify the center and find the IOC.

d) $\sum_{n=0}^{\infty} \frac{(x+3)^n}{n!}$

ex: Identify the center and find the IOC.

e) $\sum_{n=0}^{\infty} 3(x-2)^n$

Interval of Convergence

In general, the interval of convergence is...

1. c when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

2. an interval centered at c
when
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |\text{expression}|$$

3. all real numbers when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$

Radius of Convergence (ROC)

The ROC is the distance from the center to an endpoint in the IOC.

ex: Find the ROC given the IOC.

a) IOC: [4, 6) $C = 5$; $R = 1$

4 5 6

b) IOC: (17, 20) $C = 18.5$; $R = 1.5$

c) IOC: x=4 $R = 0$

d) IOC: $(-\infty, \infty)$ $R = \infty$

Differentiation Power Series

When the first term is constant, raise lower limit for f'x) by 1

ex: Find the derivative.

$f(x) =$

a) $\sum_{n=0}^{\infty} \frac{(x+3)^n}{n+1} = 1 + \frac{(x+3)}{2} + \frac{(x+3)^2}{3} + \dots$

$f'(x) = \sum_{n=1}^{\infty} \frac{n(x+3)^{n-1}}{n+1}$

$y = \frac{1}{2}(x+3)$

$y' = \frac{1}{2}$

Radius of Convergence (ROC)

The ROC is the distance from the center to an endpoint in the IOC.

ex: Find the ROC given the IOC.

a) IOC: $[4, 6)$

b) IOC: $(17, 20)$

c) IOC: $x=4$

d) IOC: $(-\infty, \infty)$

ex: Find the derivative.

b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{(2n+1)} x^{2n}}{\cancel{(2n+1)!}}$$

$\cancel{(2n+1)}(2n)!$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

ex: Find the derivative.

c) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n+1}$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n(2x)^{n-1} \cdot 2}{n+1}$$

ex: Show that the power series is a solution to the differential equation.

$$y=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!},\quad y''-y=0$$

$$y'=\sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

$$y''=\sum_{n=1}^{\infty}\frac{2n(x)^{2n-1}}{(2n)!}=\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$$

$$\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=0$$

$$\sum_{n=0}^{\infty}\frac{x^{2(n+1)-1}}{(2(n+1)-1)!}-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=0 \checkmark$$

Integration of Power Series

ex: Find the anitderivative.

a) $\sum_{n=0}^{\infty} \frac{(x+3)^n}{n+1} = 1 + \frac{x+3}{2} + \ldots\ldots$

$$C + \sum_{n=0}^{\infty} \frac{(x+3)^{n+1}}{(n+1)^2}$$

ex: Find the antiderivative.

b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

ex: Find the antiderivative.

c) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n+1}$

$C + \sum_{n=1}^{\infty} \frac{(2x)^{n+1}}{2(n+1)(n+1)}$

$C + \sum_{n=1}^{\infty} \frac{(2x)^{n+1}}{2(n+1)^2}$

$\int x^4 dx$

$\frac{x^5}{5} + c$

$\int a^n \cdot x^n$

IOC and ROC of $f(x)$, $f'(x)$, and $\int f(x)dx$

The ROC is the series obtained by differentiating or integrating a power series is the SAME as ~~the~~ that of the original series.

The IOC will have the SAME radius and center, but the endpoint convergence may differ.

$-1 < x < 1$

ex: Consider the function

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

a) Find the IOC of $f(x)$.

from earlier $[-1, 1)$

ex: Consider the function

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

b) Find the IOC of $f'(x)$.

$$\sum_{n=1}^{\infty} \frac{\cancel{n} x^{n-1}}{\cancel{n}} = \sum_{n=1}^{\infty} x^{n-1}$$

$x = -1$	$x = 1$
$\sum_{n=1}^{\infty} (-1)^{n-1}$ diverge	$\sum_{n=1}^{\infty} (1)^{n-1}$ diverge

IOC

$(-1, 1)$

ex: Consider the function

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

c) Find the IOC of $\int f(x)\,dx$.

$$C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$$

IOC

$-1 \leq x \leq 1$

$x = -1$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}$$

CONV. (A.S.T)

$x = 1$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges

D.C.T.

CONV.

$$\frac{1}{n(n+1)} \leq \frac{1}{n^2}$$

true.

ex:

What is the interval of convergence for the power series $\sum_{n=0}^{\infty}(-1)^n \frac{n+2}{n \cdot 3^n}(x-4)^n$?

(A) $-3 < x < 3$

(B) $-3 < x \leq 3$

(C) $1 < x < 7$

(D) $1 < x \leq 7$

ex:

The power series $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 5$. Which of the following must be true?

(A) The series diverges at $x = 0$.

(B) The series diverges at $x = 1$.

(C) The series converges at $x = 1$.

(D) The series converges at $x = 2$.

(E) The series converges at $x = 6$.