$$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\frac{1}{2}\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{2(2n)!}$$

$$\frac{1}{2} + \frac{1}{2}\cos 2x = \frac{1}{2} + \frac{1}{2} - \frac{(2x)^2}{2\cdot 2!} + \frac{(2x)^4}{2\cdot 4!} - \cdots$$

$$\cos x = 1 - \frac{(2x)^2}{2\cdot 2!} + \frac{(2x)^4}{2\cdot 4!} - \frac{(2x)^6}{2\cdot 6!} + \cdots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-3)^{n+1}}{5^{n+1}(n+1)^2}$$

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+2}\cdot 5^{n+1}(n+1)^2}{5^{n+2}(n+2)^2(x-3)^{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-3)(n+1)^2}{5\,(n+2)^2}\right| = \left|\frac{x-3}{5}\right| < 1$$

R

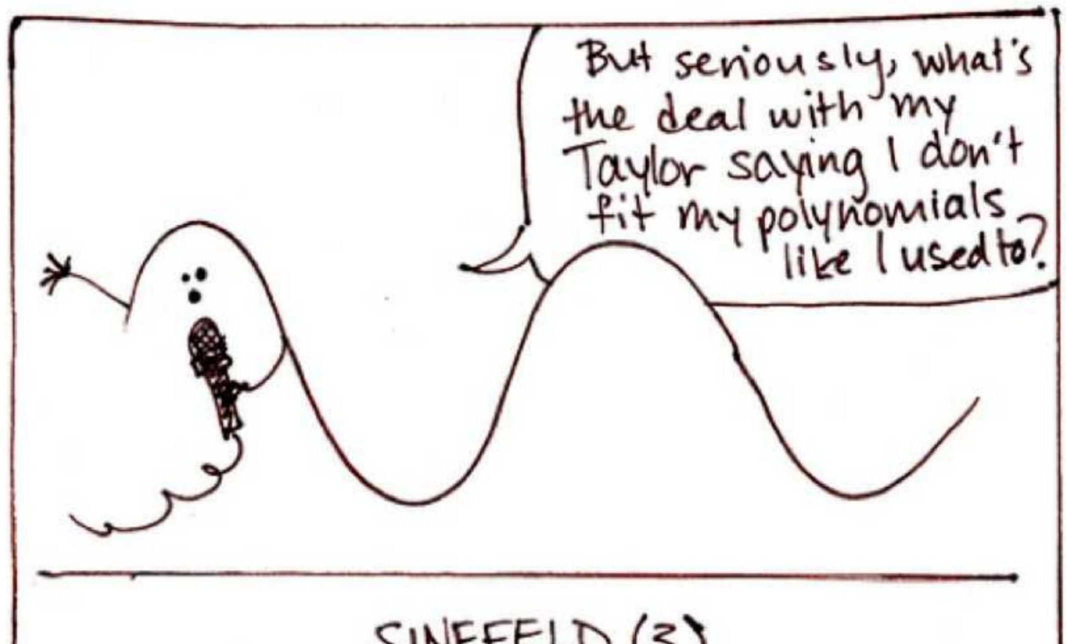
2.3)

$$f^{(n)}(3) = \frac{(-1)^n n!}{5^n n^2}$$

$$f(x) = 0 - \frac{(x-3)^1}{5} + \frac{2!(x-3)^2}{5^2 \cdot 2^2 \cdot 2!}$$

$$- \frac{3!(x-3)^3}{5^3 \cdot 3^2 \cdot 3!} + \frac{4!(x-3)^4}{5^4 \cdot 4^2 \cdot 4!} + \ldots$$

8.7 Taylor Polynomials



Taylor Polynomials

Definitions of *n*th Taylor Polynomial and *n*th Maclaurin Polynomial

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the **nth Taylor polynomial for f at c.** If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdot \cdot \cdot + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the **nth Maclaurin polynomial for f.**

**Taylor polynomials are used to approximate function values.

**Taylor polynomials only give exact function values at the center.

ex: Write the nth degree Taylor polynomial for f about x=c.

a) $f(x) = e^x$, Maclaurin, $n = 2$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$P_2(x) = 1 + x + \frac{x^2}{2!}$$

ex: Write the nth degree Taylor polynomial for f about x=c.

b) $f(x) = \ln x, \quad c = 1, \quad n = 3$

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

ex: Write the nth degree Taylor polynomial for f about x=c.

c) $f(x) = \sin 2x$, Maclaurin, $n = 5$

$$P_5(x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}$$

ex: Approximate f(1) using the 2nd degree Maclaurin polynomial for $f(x) = e^x$.

$$P_2(x) = 1 + x + \frac{x^2}{2!}$$

$$P_2(1) = 1 + 1 + \frac{1}{2} = 2\frac{1}{2}$$

ex: Approximate f(2) using the 5th degree Maclaurin polynomial for $f\left(x\right)=\sin x$.

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$P_5(2) = 2 - \frac{2^3}{3!} + \frac{2^5}{5!} = \frac{14}{15}$$

ex: What is the maximum error involved in $P_5(2)$ for $f(x) = \sin x$.

1^{ST} neglected term

$$\left| \frac{x^7}{7!} \right|$$

$$\left| \frac{2^7}{7!} \right|$$

$\approx .025396$

ex: What is the exact error involved in $P_5(2)$ for $f(x) = \sin x$.

$$\left| S - S_n \right|$$

$$\left| f(x) - P_5(x) \right|$$

$$\left| f(2) - P_5(2) \right|$$

exact error

$$\left| \sin 2 - \left(2 - \frac{8}{3!} + \frac{2^5}{5!}\right) \right| \approx 0.024$$

ex:

$\# \frac{(x-1)^n}{n!}$

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	–2	1	4
1	2	–3	3	–2
2	–1	1	4	5

Selected values of a function f and its first three derivatives are indicated in the table above. What is the third-degree Taylor polynomial for f about $x = 1$?

(A) $2 - 3x + \frac{3}{2}x^2 - \frac{1}{3}x^3$ $\quad -\frac{2}{6} \quad -\frac{2}{3!}$

(B) $2 - 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{3}(x-1)^3$

(C) $2 - 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{2}{3}(x-1)^3$

(D) $2 - 3(x-1) + 3(x-1)^2 - 2(x-1)^3$

ex:

The third-degree Taylor polynomial for the function f about $x = 0$ is $T(x) = 3 - 4x + 2x^2 - 3x^3$. Which of the following tables gives the values of f and its first three derivatives at $x = 0$?

(a)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	−8	6	−12

(b)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	−4	2	−3

(c)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	−4	4	−18

(d)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	−4	4	−9

FR 10

Let f be a function that has derivatives of all orders for all real numbers. Assume $f(1) = 3$, $f'(1) = -2$, $f''(1) = 2$, and $f'''(1) = 4$.

(a) Write the second-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(0.7)$.

(b) Write the third-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(1.2)$.

(c) Write the second-degree Taylor polynomial for f', the derivative of f, about $x = 1$ and use it to approximate $f'(1.2)$.

FR 12

Let f be the function given by $f(x) = e^{\frac{x}{2}}$.

(a) Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$ about $x = 0$.

(b) Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about $x = 0$ for $g(x) = \dfrac{e^{\frac{x}{2}} - 1}{x}$.

(c) For the function g in part (b), find $g'(2)$ and use it to show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$.