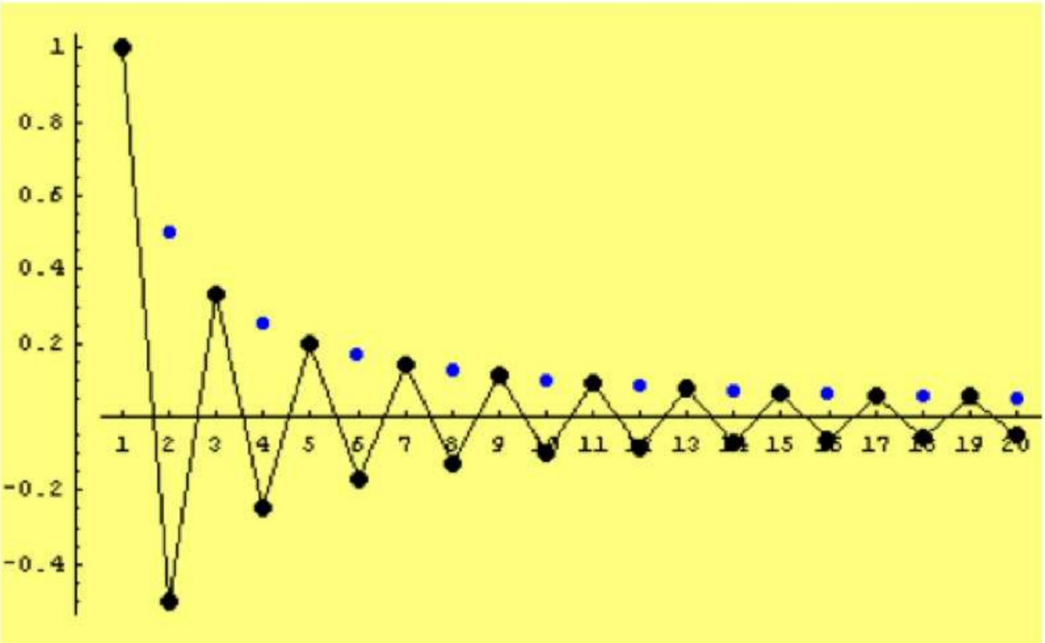
8.5 Alternating Series



Alternating Series come in many different forms...

ex: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

ex: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{3n-1}}{3^n}$

ex: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{3n+5}$

ex: $\displaystyle\sum_{n=2}^{\infty}\frac{\cos(n\pi)}{\ln n}$

etc.

Alternating Series Test (AST)

THEOREM 8.14 Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge when the two conditions listed below are met.

1. $\lim_{n \to \infty} a_n = 0$
2. $a_{n+1} \leq a_n$, for all n

*The AST is used to prove CONVERGENCE only!

*If the test fails (i.e. step 1 and/or 2 is not true), use the n^{th} term test.

ex: Determine the convergence or divergence of the series.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ $= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓ Converges by A.S.T.

$a_{n+1} \leq a_n$ ✓

ex: Determine the convergence or divergence of the series.

b) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(-2)^{n-1}} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1} n \cdot 2}{2^n} = \frac{2}{-2} + \frac{3}{4} - \frac{4}{8} + \ldots$

$(-1)^{n-1}(2)^{n-1}$

$2^{n-1} = 2^n \cdot 2^{-1}$

$\left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{-1}$

$a_{n+1} \leq a_n$ ✓

$\displaystyle\lim_{n\to\infty} \frac{2n}{2^n} = 0$ ✓

Converges by A.S.T.

ex: Determine the convergence or divergence of the series.

c) $\sum_{n=1}^{\infty} \cos(n\pi)$

$\sum_{n=1}^{\infty} (-1)^n \cdot 1$

cannot use A.S.T.

$a_n = 1$

$\lim_{n \to \infty} \cos(n\pi) = \text{dne} \neq 0$

diverges by n^{th} term test

ex: Determine the convergence or divergence of the series.

d) $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n}$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (harmonic)

Condition & Absolute Convergence

THEOREM 8.16 Absolute Convergence

If the series $\Sigma\, |a_n|$ converges, then the series $\Sigma\, a_n$ also converges.

Definitions of Absolute and Conditional Convergence

1. The series $\Sigma\, a_n$ is **absolutely convergent** when $\Sigma\, |a_n|$ converges. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
2. The series $\Sigma\, a_n$ is **conditionally convergent** when $\Sigma\, a_n$ converges but $\Sigma\, |a_n|$ diverges. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

*Test the absolute value of the series 1st, unless you can identify the series is divergent.

*If the original series diverges, just write "divergent"

ex: Determine the convergence or divergence of the series. Classify any convergent series as absolutely or conditionally convergent.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (alternating harmonic series)

proven : convergent by A.S.T.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (pseries, $p=1$)

Converges conditionally

ex: Determine the convergence or divergence of the series. Classify any convergent series as absolutely or conditionally convergent.

b) $\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n$ $= \sum_{n=1}^{\infty}\left|(-1)^n\left(\frac{1}{3}\right)^n\right|$

$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$

converges

geo.

$r=\frac{1}{3}<1$

ex: Determine the convergence or divergence of the series. Classify any convergent series as absolutely or conditionally convergent.

c) $\sum_{n=0}^{\infty} \left|\frac{\sin n}{5^n}\right|$

$\left|\frac{\sin n}{5^n}\right| \leq \left|\frac{1}{5^n}\right|$ ✓

Converges by D.C.T.

$\sum_{n=0}^{\infty} \frac{1}{5^n}$

Converges absolutely

$\lim_{n\to\infty} \frac{\sin n}{\cancel{5^n}} \cdot \frac{\cancel{5^n}}{1}$

L.C.T. doesn't work

Symbol Recognition

a_n	n^{th} term
S_n	n^{th} partial sum (sum of the 1^{st} n terms)
S	sum of all terms
R_n	error involved in S_n

Exact Error

$$|S - S_n|$$

$$\text{error} = |\text{actual - approximation}|$$

$$^*\text{error} \geq 0$$

$$\boxed{R_n = |S - S_n|}$$

Error Bound (Max Error)

An error bound is a value that is LARGER than the actual error.

$$\boxed{\text{error bound} \geq R_n}$$

There are two types of error bounds we will discuss:

- Alternating Series Remainder Theorem
- Lagrange Error Bound

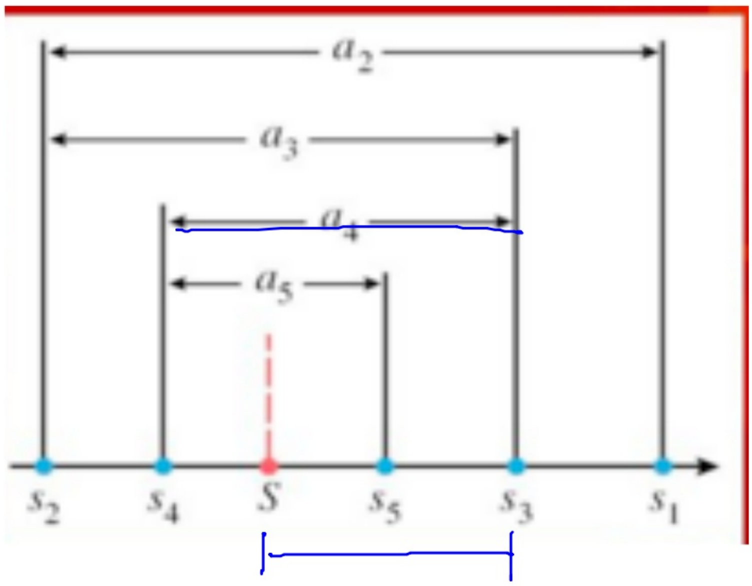
Alternating Series Remainder Theorem

THEOREM 8.15 Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \le a_{N+1}.$$

*Essentially, if an alternating series is CONVERGENT then the error bound (or max error) is the "1st neglected term."



a_2
a_3
a_4
a_5
s_2
s_4
S
s_5
s_3
s_1

ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

a) Find S_4 $= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} = \frac{5}{8}$

b) Find the maximum error involved in S_4. $\frac{1}{5!} = \frac{1}{120}$

ex: Determine the minimum number of terms needed to approximate the sum of the series with an error less than 0.001. $\frac{1}{1000}$

less than .001

1st neglected term

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} = \underbrace{\frac{1}{4} - \frac{1}{32} + \frac{1}{192}}_{3} - \frac{1}{1024}$$

ex: Approximate the sum such that the error is less than 0.001.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

$$S_5 = 1 - \frac{1}{16} + \frac{1}{81} - \frac{1}{256} + \frac{1}{625}$$

$$\approx .948$$

ex: If S_5 is used to approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$, find an interval in which the sum must lie.

$$|S - S_5| \leq \frac{1}{1296}$$

$$|S - .948| \leq \frac{1}{1296}$$

$$-\frac{1}{1296} \leq S - .948 \leq \frac{1}{1296}$$

$$.947 \leq S \leq .948$$

ex:

The alternating series test can be used to show convergence of which of the following alternating series?

I. $4 - \frac{1}{9} + 1 - \frac{1}{81} + \frac{1}{4} - \frac{1}{729} + \frac{1}{16} - \cdots + a_n + \cdots$, where $a_n = \begin{cases} \frac{8}{2^n} & \text{if } n \text{ is odd} \\ -\frac{1}{3^n} & \text{if } n \text{ is even} \end{cases}$

$a_1 \quad a_2 \quad a_3 \quad a_4$

II. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots + a_n + \cdots$, where $a_n = \frac{(-1)^{n+1}}{n}$

III. $\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \frac{6}{11} - \frac{7}{13} + \frac{8}{15} - \cdots + a_n + \cdots$, where $a_n = (-1)^{n+1}\frac{n+1}{2n+1}$

(A) ~~I only~~
(B) II only
(C) III only
(D) I and II only
(E) I, II, and III

ex:

Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ is true?

(A) The series converges absolutely.

(B) The series converges conditionally.

(C) The series converges but neither conditionally nor absolutely.

(D) The series diverges.

Fill in your convergence chart...

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
AST for convergence	$\sum_{n=\#}^{\infty} (-1)^n a_n$	1 $a_{n+1} \le a_n$ 2 $\lim_{n \to \infty} a_n = 0$		use nth term test if AST fails