8.3 The Integral Test & P-Series

The Integral Test

THEOREM 8.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

* The integral value is NOT the sum of the series.

ex: Determine the convergence or divergence of the series.

a) $\sum_{n=1}^{\infty} 2ne^{-n^2}$

P ✓

C ✓

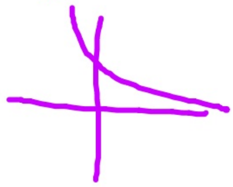
D ✓

Series converges by Integral Test

$$\lim_{b\to\infty}\int_1^b 2xe^{-x^2}\,dx$$

$$\lim_{b\to\infty}\left(-e^{-x^2}\right)\Big|_1^b$$

e^{-10000}



$$\lim_{b\to\infty}\left(-e^{-b^2}+e^{-1}\right)$$

$$0+\frac{1}{e}=\frac{1}{e}$$

ex: Determine the convergence or divergence of the series.

b) $\sum_{n=1}^{\infty} \frac{1}{n}$

C ✓
P ✓
D ✓

$$\lim_{b \to \infty} \int_1^b \frac{1}{x}\,dx$$

$$\lim_{b \to \infty} \ln|x| \Big|_1^b$$

$$\lim_{b \to \infty} \ln b - 0 = \infty$$

Series diverges by Integral Test

p-Series

$\frac{1}{n}$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \qquad p\text{-series}$$

$\frac{1}{n^2}$

$\frac{1}{n^{1/2}}$

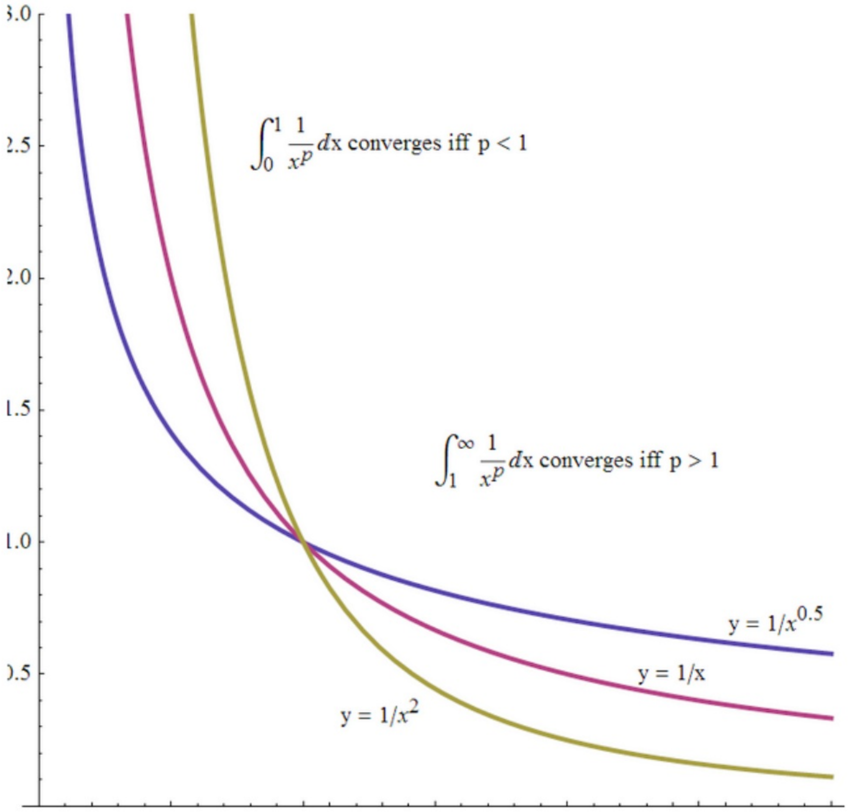
THEOREM 8.11 Convergence of *p*-Series

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

converges for $p > 1$, and diverges for $0 < p \leq 1$.

A helpful graph



The Harmonic Series

Harmonic Series:

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$

General Harmonic Series:

$\frac{2}{3n+1}$

$$\sum_{n=1}^{\infty} \frac{a}{bn+c}; \quad a,b,c \in R$$

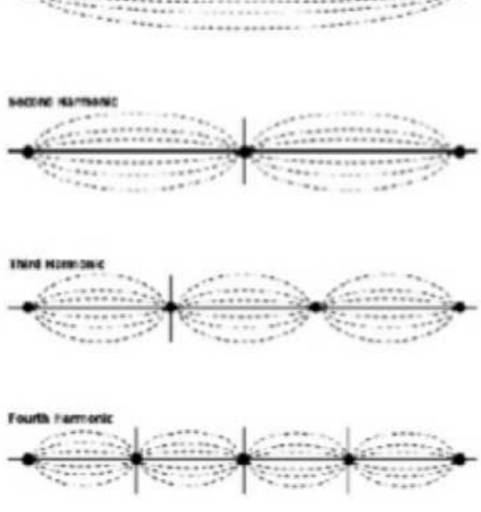
These series DIVERGE!

FUN FACT:

Sound and the Harmonic Series

'Natural' sound producing instruments (pipes, 'strings') have a tendency to produce multiple sounds in proportion to their length:

1, ½, ⅓, ¼, ⅕, ⅙, etc.



ex: Determine the convergence or divergence of the series.

a) $\sum_{n=1}^{\infty} \frac{5}{\sqrt[3]{n}}$ divergent $p = 1/3$ $0 < p \leq 1$
by
p-series test

b) $\sum_{n=1}^{\infty} \left(\frac{7}{n^6} - \frac{2}{n^3} \right)$ convergent by p-series both p's > 1
$p = 6, 3$

ex: Determine the convergence or divergence of the series.

c) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergence (geo.)

$|r| < 1$

$\left(\frac{1}{2}\right)^n$ $r = 1/2$

d) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

p-series test

$p > 1$

ex: Determine the convergence or divergence of the series. Show the work that leads to your conclusion. Use ANY TEST and state the test used.

a) $\sum_{n=1}^{\infty} \ln(n)$ Div. nth term

d) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ integral diverges

b) $\sum_{n=1}^{\infty} (1.001)^n$ Div. $r = 1.001$ geo

e) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ $\frac{1}{n^2}$ p-series $p=2$ conv.

c) $\sum_{n=1}^{\infty} \cos n$ Div. nth term

f) $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \ldots$ $\frac{1}{n^2+1}$

ex: Determine the convergence or divergence of the series. Show the work that leads to your conclusion. Use ANY TEST and state the test used.

a) $\sum_{n=1}^{\infty} \ln(n)$

ex: Determine the convergence or divergence of the series. Show the work that leads to your conclusion. Use ANY TEST and state the test used.

b) $\sum_{n=1}^{\infty}(1.001)^n$

ex: Determine the convergence or divergence of the series. Show the work that leads to your conclusion. Use ANY TEST and state the test used.

c) $\sum_{n=1}^{\infty} \cos n$

ex: Determine the convergence or divergence of the series. Show the work that leads to your conclusion. Use ANY TEST and state the test used.

divergent
integral test

d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$$\lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \ln|\ln x|\Big|_2^b$$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$$\lim_{b\to\infty} \ln|\ln b| - 0$$

$$\infty$$

ex: Determine the convergence or divergence of the series. Show the work that leads to your conclusion. Use ANY TEST and state the test used.

e) $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$

ex: Determine the convergence or divergence of the series. Show the work that leads to your conclusion. Use ANY TEST and state the test used.

f) $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\ldots$

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \arctan \Big|_1^b$$

$$\frac{\pi}{2}-\frac{\pi}{4}$$

$$\frac{\pi}{4}$$

ex:

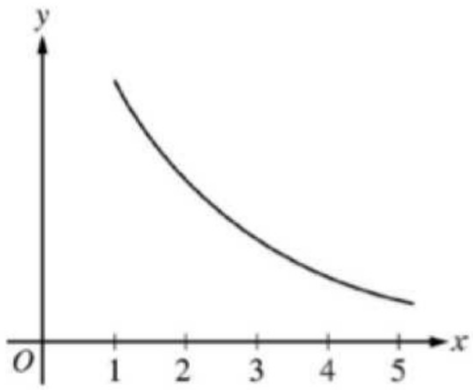
For $x \geq 1$, the continuous function g is decreasing and positive. A portion of the graph of g is shown For $n \geq 1$, the nth term of the series $\sum_{n=1}^{\infty} a_n$ is defined by $a_n = g(n)$. If $\int_1^{\infty} g(x)\, dx$ converges to 8, which of the following could be true?

(A) $\sum_{n=1}^{\infty} a_n = 6$

(B) $\sum_{n=1}^{\infty} a_n = 8$

(C) $\sum_{n=1}^{\infty} a_n = 10$

(D) $\sum_{n=1}^{\infty} a_n$ diverges



Graph of g

Fill in your convergence chart...

$\# \in \mathbb{W}$

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Integral cont., pos. decr.	$\sum_{n=\#}^{\infty} a_n$ $a_n = f(n) \geq 0$	Lim finite value	∞ DNE	

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
p-series	$\sum_{n=\#}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	