8.2 Introduction to Series

series - the sum of a sequence

$$\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots$$

Summation Notation

$$\sum_{n=1}^{i} a_n = a_1 + a_2 + a_3 + \ldots a_i$$

n = index of summation
i = upper limit
a_n = summand

ex: Find the sum.

a) $\sum_{n=1}^{3} 2^n = 2^1 + 2^2 + 2^3 = 14$

b) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \ldots$

$r = \frac{2}{3}$

$$S = \frac{a_1}{1-r} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

Partial Sums - S_n

$$
\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4 \\
S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n
\end{aligned}
$$

ex: $\sum_{n=1}^{\infty} \frac{4}{n(n+1)}$

Find the indicated partial sum.

a) S_5 $= \frac{10}{3}$

b) S_{21} $= 3.818$

$\frac{42}{11}$

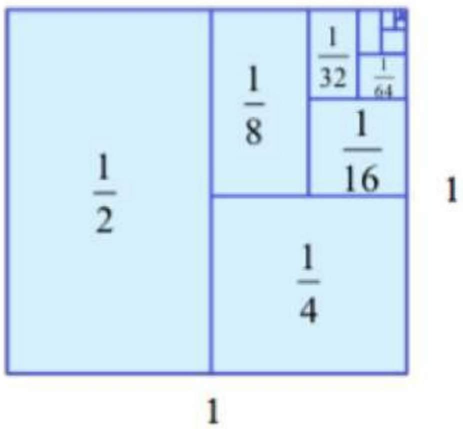
REVIEW: Convergent and Divergent SEQUENCES

Given a sequence: a_n

CONVERGENT WHEN: $\lim_{n \to \infty} a_n = $ finite number

DIVERGENT WHEN: $\lim_{n \to \infty} a_n = $ dne or ∞

Convergent and Divergent Infinite Series



$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=1$$

If the sum of an infinite series is finite, the series is said to <u>converge</u> (i.e. approaches a limiting value).

Likewise, if the sum of an infinite series is infinite, the series is said to <u>diverge</u> (i.e. the limit is infinite or DNE).

Convergent and Divergent Series

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is

$$S_n = a_1 + a_2 + \cdot \cdot \cdot + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$$S = a_1 + a_2 + \cdot \cdot \cdot + a_n + \cdot \cdot \cdot \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series **diverges.**

*
In other words, a series converges iff

$$\lim_{n \to \infty} S_n = \text{ finite quantity}$$

ex: Use the definition of series convergence to determine if the series converges or diverges. Find the sum, if possible.

a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$

$S_n = \frac{2^n - 1}{2^n}$

$S_n = \frac{1 \cdot 2^n - 1}{1 \cdot 2^n}$

$\lim_{n\to\infty} S_n = 1$ Sum!

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}}$

$S = 1$

ex: Use the definition of series convergence to determine if the series converges or diverges. Find the sum, if possible.

b) $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$S_n = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\dots$

$= \left(1-\frac{1}{n+1}\right)$

$\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$

converges

sum $=1$

In many cases S_n is difficult to find. To quickly and easily determine the convergence or divergence of a series we will often use shortcut tests.

Summary of Convergence Tests

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Geometric				
nth-Term				
p-Series				
Integral				
Alternating				
Direct Comparison				
Limit Comparison				
Ratio				
Root				

Geometric Series

$$\boxed{\sum_{n=\#}^{\infty} ar^n, \quad a \neq 0, \quad r \neq 0, \quad \# \in W}$$

$W = \{0, 1, 2 \ldots\}$

r: common ratio

*The series...

- converges when $|r| < 1$
- diverges when $|r| \geq 1$

If $|r| < 1, \; S = \dfrac{a}{1-r}$

ex: Determine the convergence or divergence of the series. Then, find the sum if possible.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n}$ geometric $r = \frac{1}{2}$ $|r| < 1$ convergent

$$S = \frac{3}{1 - \frac{1}{2}}$$

$S = 6$

b) $\displaystyle\sum_{n=22}^{\infty} e^n$ geometric $r = e$ $|r| \geq 1$ diverge ex: Determine the convergence or divergence of the series. Then, find the sum if possible.

c) $\sum_{n=1}^{\infty} 3^{2n} = \sum_{n=1}^{\infty} 9^n$

geometric $r=9$ $|r| \geq 1$

diverges

ex: Find all values of x for which the series converges. For these values of x, write the sum of the series as a function of x.

$|r| < 1$

$$\sum_{n=0}^{\infty} \left(\frac{2}{x}\right)^n$$

$x > 2, x < -2$

$\boxed{|x| > 2}$ *

$-2 \quad 2$

$$S = \frac{1}{1 - \frac{2}{x}}$$

$$S = \frac{x}{x-2}, \quad |x| > 2$$

Series and Convergence

THEOREM 8.8 Limit of the *n*th Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

nth Term Test for DIVERGENCE

THEOREM 8.9 *n*th-Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

BE CAREFUL: The converse is not always true!!!

If $\lim_{n\to\infty} a_n = 0$, then the nth term test is inconclusive

USE ANOTHER TEST

ex: Determine the convergence or divergence of the series. Show the work that leads to your conclusion and state the test used.

a) $\sum_{n=2}^{\infty} \ln n$ $\quad \lim_{n\to\infty} \ln n = \infty \neq 0$ Divergent; n^{th} term test

b) $\sum_{n=1}^{\infty} 2^n$ $\quad \lim_{n\to\infty} 2^n = \infty \neq 0$ Divergent; n^{th} term test

c) $\sum_{n=3}^{\infty} \frac{n+1}{2n-1}$ $\quad \lim_{n=3} \left(\frac{n+1}{2n-1}\right) = \frac{1}{2} \neq 0$ Divergent n^{th} term test ex: Determine the convergence or divergence of the series. Show the work that leads to your conclusion and state the test used.

d) $\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^{3n}$

$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{3n} = e^6 \neq 0$

Divergent by n^{th} term test

e) $\sum_{n=1}^{\infty} \arctan n$

✓

f) $\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^{n}$ 2.7 3.14

geometric $|r|<1$

converges

ex:

What is the sum of the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{e^{n+1}}$?

$\left(\frac{-2}{e}\right)^n \cdot \frac{1}{e}$

(A) $\frac{-2}{e^2 - 2e}$ (B) $\frac{-2}{e^2 + 2e}$ (C) $\frac{-2}{e+2}$ (D) $\frac{e}{e+2}$ (E) The series diverges.

$$S = \frac{\frac{-2}{e^2}}{1 + \frac{2}{e}} = B$$

ex:

The infinite series $\sum_{k=1}^{\infty} a_k$ has nth partial sum $S_n = \frac{n}{3n+1}$ for $n \geq 1$. What is the sum of the series $\sum_{k=1}^{\infty} a_k$?

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{3}{2}$ (E) The series diverges.

Fill in your convergence chart...

NEW

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Geometric	$\sum_{n=\#}^{\infty} a \cdot r^n$	$\lvert r \rvert < 1$	$\lvert r \rvert \geq 1$	$S = \frac{1^{st} \text{ term}}{1-r}$

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
*n*th-Term	$\sum_{n=\#}^{\infty} a_n$		$\lim_{n \to \infty} a_n \neq 0$	

Just for fun...

Proof: For geometric series where $|r| < 1,\ S = \dfrac{a}{1-r}$

Start with a geometric sequence.

$$S = a + ar + ar^2 + ar^3 + ar^4 + \ldots$$

Multiply both sides by r.

$$Sr = ar + ar^2 + ar^3 + ar^4 + \ldots$$

Subtract the two series...

$$S - Sr = a$$
$$S(1 - r) = a$$
$$S = a/(1 - r)$$