7.) $f(x) = \cos(x^{3/2})$ $\qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

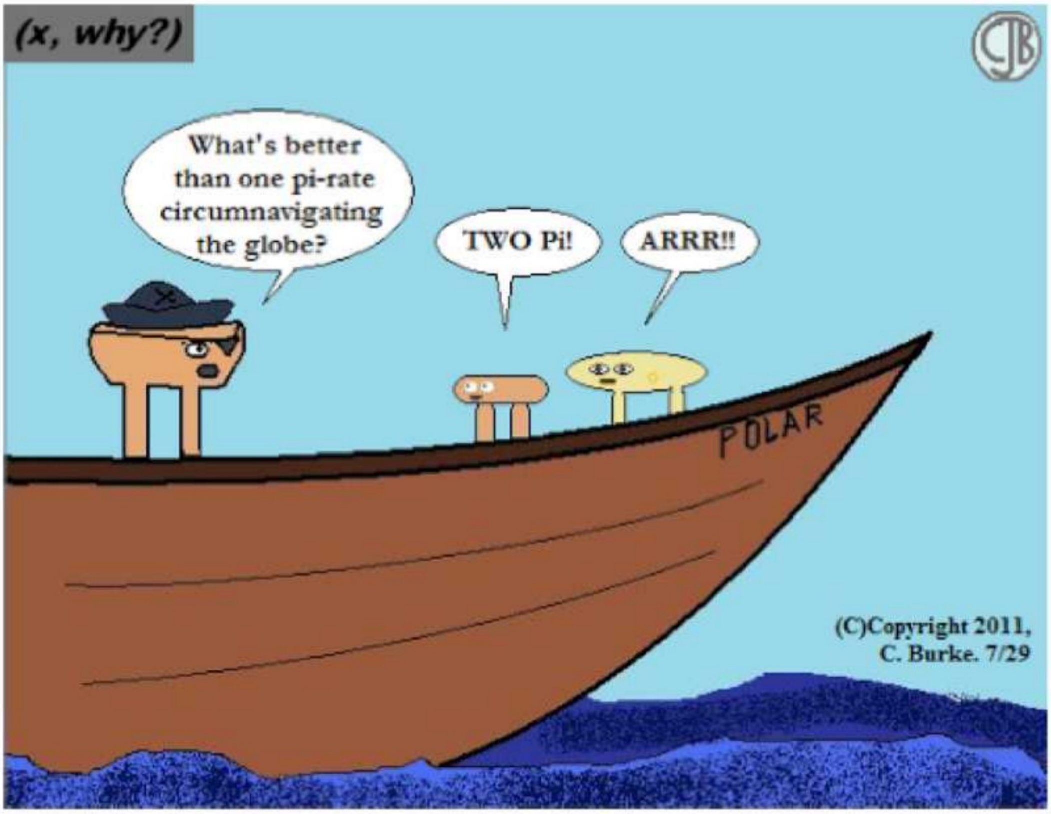
$$\cos x^{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!}$$

$$f(x) = 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \frac{x^9}{6!} + \dots$$

$$+ \frac{(-1)^n x^{3n}}{(2n)!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$$

8.10 Taylor & Maclaurin Series - Cont.



*See printout.

ex: Write the first four nonzero terms and the general term of the Taylor series expansion of f(x) about x=c.

a) $f(x) = \arctan(x^2), \quad c = 0$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\arctan(x^2) = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} + \frac{x^{14}}{7} + \dots + \frac{(-1)^n x^{4n+2}}{2n+1} + \dots$$

ex: Write the first four nonzero terms and the general term of the Taylor series expansion of f(x) about x=c.

b) $f\left(x\right)=\sin^2 x, \quad c=0$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots$$

$$-\frac{1}{2}\cos 2x = -\frac{1}{2} + \frac{(2x)^2}{2\cdot 2!} - \frac{(2x)^4}{2\cdot 4!} + \frac{(2x)^6}{2\cdot 6!} + \ldots$$

$$\frac{1}{2} - \frac{1}{2}\cos 2x = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{(2x)^2}{2\cdot 2!} - \frac{(2x)^4}{2\cdot 4!} + \frac{(2x)^6}{2\cdot 6!} - \frac{(2x)^8}{2\cdot 8!} + \ldots$$

$$\ldots + \frac{(-1)^n x^{2n+2}}{2(2n+2)!} + \ldots$$

ex: Write the first four nonzero terms and the general term of the Taylor series expansion of f(x) about x=c.

c) $\int_0^x \left(\cos\frac{t}{3} - 1\right)dt \quad c=0$

$$\int_0^x \left(\cancel{1} - \frac{\left(\frac{t}{3}\right)^2}{2!} + \frac{\left(\frac{t}{3}\right)^4}{4!} - \frac{\left(\frac{t}{3}\right)^6}{6!} + \ldots \cancel{-1}\right)dt$$

$$\int_0^x \left(-\frac{t^2}{3^2 \cdot 2!} + \frac{t^4}{3^4 \cdot 4!} - \frac{t^6}{3^6 6!} + \ldots\right)dt$$

$$\left.\frac{-t^3}{3 \cdot 3^2 2!} + \frac{t^5}{5 \cdot 3^4 4!} - \frac{t^7}{7 \cdot 3^6 6!} + \ldots\right]_0^x$$

$$\frac{-x^3}{3^2 \cdot 2! \cdot 3} + \frac{x^5}{3^4 \cdot 4! \cdot 5} - \frac{x^7}{3^6 \cdot 6! \cdot 7} + \frac{x^9}{3^8 \cdot 8! \cdot 9} + \cdots$$

$$+ \frac{(-1)^{n+1} \, x^{2n+3}}{3^{2n+2} \underbrace{(2n+2)!(2n+3)}_{(2n+3)!}}$$

ex: Use a Maclaurin series to evaluate the limit.

$$\lim_{x \to 0} \arctan\left(x^2\right)$$

$$\lim_{x \to 0} \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \dots \right) = 0$$

ex: Approximate the integral value with an error less than 0.01. Justify your answer.

$$\int_0^1 \arctan\left(x^2\right)dx$$

$$\int_0^1 \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \dots\right)dx$$

$$\left.\frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55} - \frac{x^{15}}{105} + \dots \right]_0^1$$

Actual
.298

$$\underbrace{\frac{1}{3} - \frac{1}{21} + \frac{1}{55}} - \frac{1}{105} + \dots$$

estimate
.304

less than $\frac{1}{100}$

ex: Consider the function:

$$f(x) = -\frac{1}{2!} + \frac{x}{3!} - \frac{x^2}{4!} + \dots + \frac{(-1)^n x^n}{(n+2)!} + \dots$$

Find the indicated derivatives.

a) $f'(0)$ $\quad f'(x) = 0 + \frac{1}{3!} - \frac{2x}{4!} + \dots$

$f'(0) = \frac{1}{3!}$

ex: Consider the function:

$$\frac{x^3}{5!}$$

$$f(x) = -\frac{1}{2!} + \frac{x}{3!} - \frac{x^2}{4!} + \ldots + \frac{(-1)^n x^n}{(n+2)!} + \ldots$$

Find the indicated derivative.

b) $f''(0) = -\frac{2}{4!}$

$f'(x) = \frac{1}{3!} - \frac{2x}{4!} + \frac{3x^2}{5!}$

$f''(x) = \frac{-2}{4!} + \frac{6x}{5!} + \ldots$

$f''(0) = -\frac{2}{4!} + 0 \ldots$

ex: Consider the function:

$$f(x) = -\frac{1}{2!} + \frac{x}{3!} - \frac{x^2}{4!} + \ldots + \frac{(-1)^n x^n}{(n+2)!} + \ldots$$

Find the indicated derivative.

c) $f^{(21)}(0)$

$n = 21$

formula $\qquad$ Actual

$$\frac{f^n(c)(x-c)^n}{n!} = \frac{(-1)^n x^n}{(n+2)!}$$

$$\frac{f^{(21)}(0)\cancel{x^{21}}}{21!} = \frac{(-1)^{21}\cancel{x^{21}}}{23!}$$

$$f^{(21)}(0) = \frac{(-1)^{21}21!}{23!} = \frac{-21!}{23!}$$

ex: Consider the function:

$$f(x) = -\frac{1}{2!} + \frac{x}{3!} - \frac{x^2}{4!} + \ldots + \frac{(-1)^n x^n}{(n+2)!} + \ldots$$

Find the indicated derivative.

d) $f^{(19)}(7)$ not possible; center is 0

Finding the derivative at the center works because all of the terms (except for one) become 0. For any other value, this won't happen and you would have an infinite number of terms with an x.

ex: Consider the function:

$$f(x) = x^2 + \frac{x^3}{3} + \frac{x^4}{9} + \frac{x^5}{27} + \ldots + \frac{x^{n+2}}{3^n} + \ldots$$

$f'(x) = 2x + x^2 + \frac{4x^3}{9} + \ldots$

$f''(x) = 2 + 2x + \frac{12x^2}{9} + \ldots$

a) Find $f'(0) = 0$

b) Find $f''(0) = 2$

c) Find $f^{(17)}(0)$

Formula   Actual $n=15$

$$\frac{f^{(17)}(0)\,\cancel{x^{17}}}{17!} = \frac{\cancel{x^{15+2}}}{3^{15}}$$

$$f^{(17)}(0) = \frac{17!}{3^{15}}$$

ex: Consider the function:

$$f(x) = x^2 + \frac{x^3}{3} + \frac{x^4}{9} + \frac{x^5}{27} + \ldots + \frac{x^{n+2}}{3^n} + \ldots$$

d) Does f(x) have a relative maximum, relative minimum or neither at x=0? Justify your answer.

2nd deriv test.

$f'(c) = 0$

* $f''(c) > 0$ rel. min

$f''(c) < 0$ rel. max.

$f''(c) = 0$

$f'(0) = 0$

$f''(0) = 2$

rel. min by 2nd Deriv. Test.

ex: The nth derivative of f(x) at x=1 is given by

$$f^n(1) = \frac{(n+1)!}{2^n}, \quad n \geq 1 \text{ and } f(1)=1.$$

a) Write the first four nonzero terms and the general term of the Taylor series expansion of f(x) about x=1. $C = 1$

$$f(x) = 1 + \frac{2!}{2}(x-1)^1 + \frac{3!}{2^2 \cdot 2!}(x-1)^2 + \frac{4!}{3!\,2^3}(x-1)^3 + \ldots$$

$$= 1 + (x-1)^1 + \frac{3(x-1)^2}{2^2} + \frac{4(x-1)^3}{2^3} + \ldots \frac{(n+1)(x-1)^n}{2^n} + \ldots$$

ex: The nth derivative of f(x) at x=1 is given by

$$f^{n}(1)=\frac{(n+1)!}{2^{n}},\quad n\geq 1 \text{ and } f(1)=1.$$

b) Find the radius of convergence for the Taylor series for f about x=1. Show the work that leads to your conclusion.

$$\sum_{n=0}^{\infty}\frac{(n+1)(x-1)^n}{2^n}=\sum_{n=0}^{\infty}(n+1)\left(\frac{x-1}{2}\right)^n \qquad \underline{R=2}$$

$$\lim_{n\to\infty}\left|\frac{(n+2)(x-1)^{n+1}\cdot 2^n}{2^{n+1}(n+1)(x-1)^n}\right|=\lim_{n\to\infty}\left|\frac{(x-1)(n+2)}{2(n+1)}\right|=\left|\frac{x-1}{2}\right|<1$$

FR 7

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

FR 9

The Maclaurin series for $f(x)$ is given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$

(a) Find $f'(0)$ and $f^{(17)}(0)$.

(b) For what values of x does the given series converge? Show your reasoning.

(c) Let $g(x) = x\,f(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.