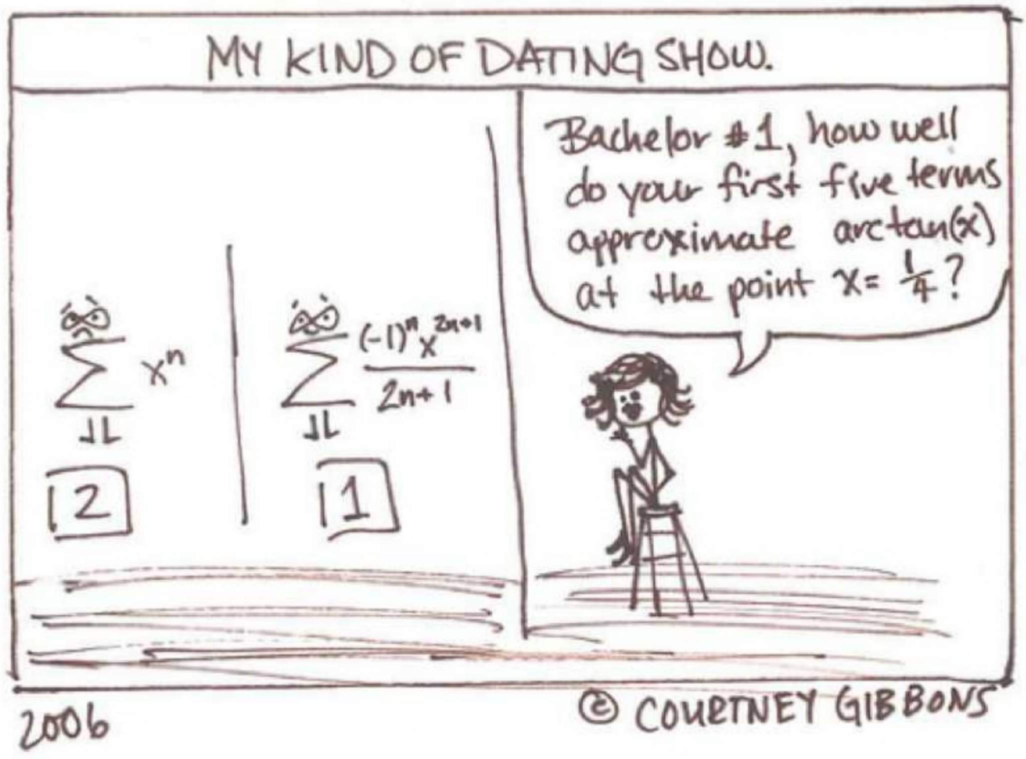
8.10 Taylor & Maclaurin Series



Taylor & Maclaurin Series

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdot \cdot \cdot$$

is called the **Taylor series for f at c.** Moreover, if $c = 0$ then the series is the **Maclaurin series for f.**

8.10

ex: Write the first four nonzero terms and the general term of the Maclaurin series expansion of f(x). State the IOC.

$(-\infty, \infty)$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$

a) $f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x \quad f'(0) = 1$

$f''(x) = e^x \quad f''(0) = 1$

$f'''(x) = e^x \quad f'''(0) = 1$

Ratio Test

$\lim_{n\to\infty} \left(\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right) = 0$

$f(x) = 1 + 1 \cdot x + \frac{1 \cdot x^2}{2!} + \frac{1 \cdot x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$

ex: Write the first four nonzero terms and the general term of the Maclaurin series expansion of f(x). State the IOC.

b) $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$f(0) = 0$

$f'(0) = 1$ ✓

$f''(0) = 0$

$f'''(0) = -1$

$f^{(4)}(0) = 0$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$(-\infty, \infty)$

$$f(x) = 1 \cdot x - \frac{1 \cdot x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

ex: Write the first four nonzero terms and the general term of the Maclaurin series expansion of f(x). State the IOC.

c) $f(x) = \cos x$

$$g(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$g'(x) = f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

ex: Write the first four nonzero terms and the general term of the Maclaurin series expansion of f(x). State the IOC.

$\frac{1}{1+x^2}$

d) $f(x) = \tan^{-1} x$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$x=-1$	$x=1$
$\sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+1}$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$
CONV. A.S.T.	CONV. A.S.T.

$[-1, 1]$

$\boxed{\text{p. 618}}$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$f''(x) = -1(1+x^2)^{-2}(2x)$

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \frac{(-1)^n x^{2n+1}}{2n+1} + \ldots$$

ex: Write the first four nonzero terms and the general term of the Maclaurin series expansion of f(x). State the IOC.

$(-1, 1)$

e) $f\left(x\right)=\dfrac{1}{1-x}$

$f(x) = 1 + x + x^2 + x^3 + \ldots x^n + \ldots$

ex: Write the first four nonzero terms and the general term of the Maclaurin series expansion of f(x). State the IOC.

f) $f(x) = \sin 2x$

$g(x) = \sin x$

$$g(2x) = f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

$$f(x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} + \dots$$

ex: Write the first four nonzero terms and the general term of the Maclaurin series expansion of f(x). State the IOC.

$$g(x) = e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

g) $f(x) = xe^x$

$$f(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \dots \frac{x^{n+1}}{n!} + \dots$$

ex: Write the first four nonzero terms of the Taylor series expansion of f(x) about x=c.

a) $f(x) = \sin x, \quad c = \dfrac{\pi}{4}$

ex: Write the first four nonzero terms of the Taylor series expansion of f(x) about x=c.

b) $f(x)=\ln x, \quad c=1$

$$f(x)=f'(1)(x-1)+\frac{f''(1)(x-1)^2}{2!}+\frac{f'''(1)(x-1)^3}{3!}+\frac{f^{(4)}(1)(x-1)^4}{4!}$$

$f'(x)=\frac{1}{x} \quad f'(1)=1$

$f''(x)=-\frac{1}{x^2} \quad f''(1)=-1$

$f'''(x)=+\frac{2}{x^3} \quad f'''(1)=2$

$f^{(4)}(x)=-\frac{6}{x^4} \quad f^{(4)}(1)=-6$

$$f(x)=(x-1)-\frac{(x-1)^2}{2!}+\frac{2(x-1)^3}{3!}-\frac{6(x-1)^4}{4!}+\ldots+\frac{(-1)^n n!(x-1)^{n+1}}{(n+1)!}+\ldots$$

Fill in your "Basic 5" Chart

"Basic 5"

	Function		IOC	Center
1.	$f(x) = e^x$			
2.	$f(x) = \sin x$			
3.	$f(x) = \cos x$			
4.	$f(x) = \arctan x$			
5.	$f(x) = \frac{1}{1-x}$			