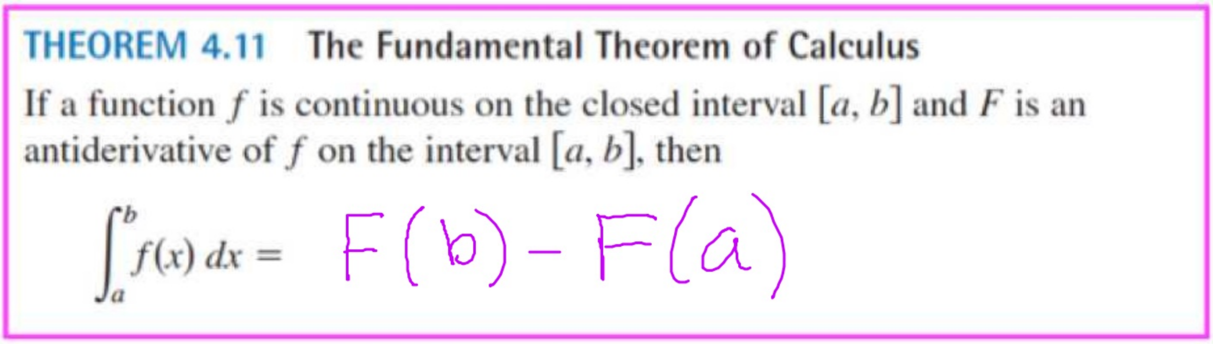
7.8 Improper Integrals

Review:

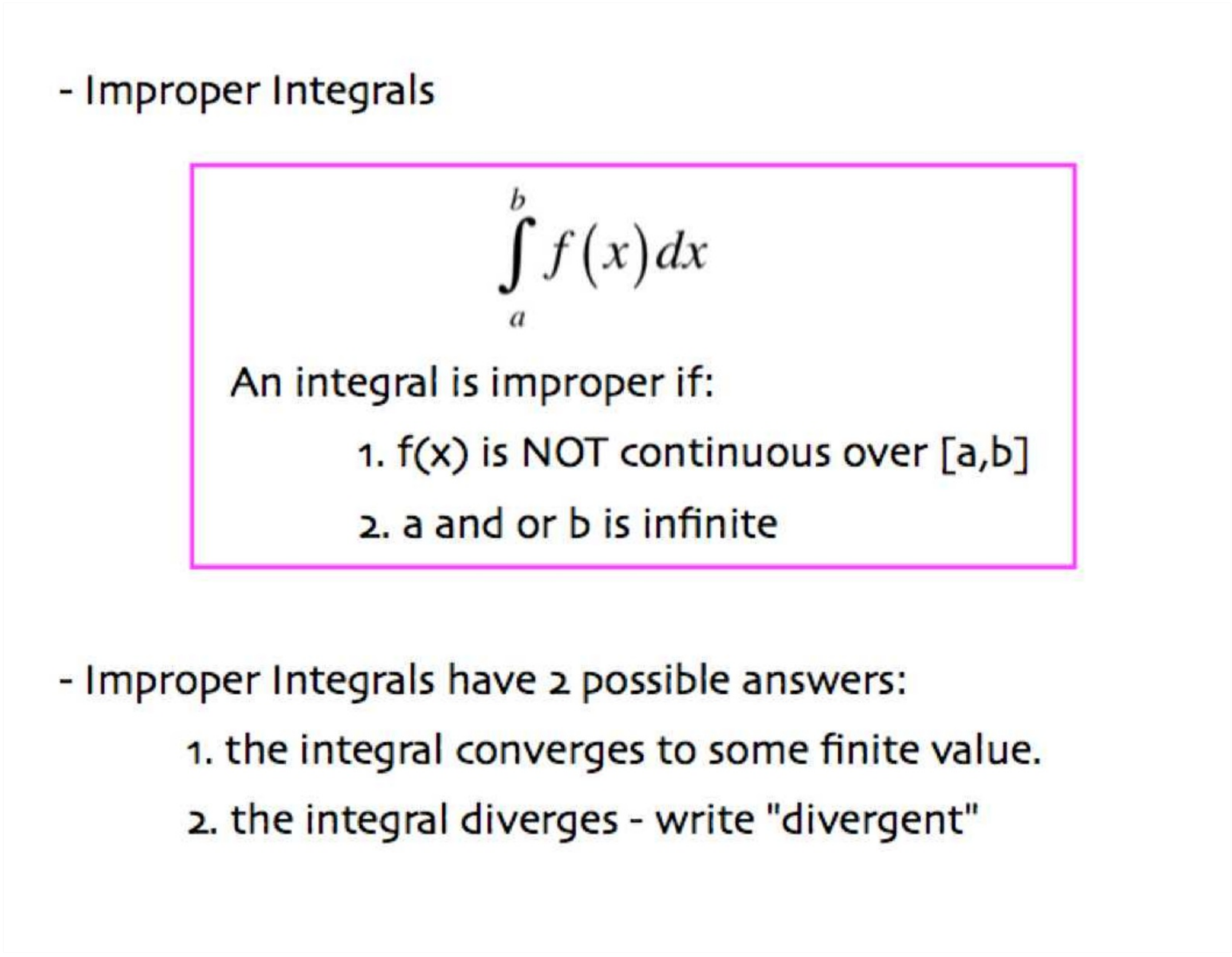


THEOREM 4.11 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

*f(x) must be continuous
*a and b must be finite



- Improper Integrals

$$\int_{a}^{b} f\left(x\right)dx$$

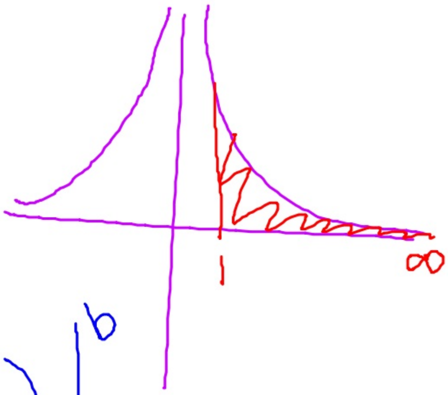
An integral is improper if:

1. f(x) is NOT continuous over [a,b]
2. a and or b is infinite

- Improper Integrals have 2 possible answers:
  1. the integral converges to some finite value.
  2. the integral diverges - write "divergent"

ex: Evaluate.

a) $\int_1^\infty \frac{1}{x^2}dx$



$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_1^b$$

$$= \lim_{b\to\infty} \left(-\overset{\to 0}{\frac{1}{b}} + 1\right) = 1$$

$$S = \frac{a_1}{1-r}$$

$$S = \frac{3}{1-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} 3\cdot\left(\underset{\uparrow}{\frac{1}{2}}\right)^{n}$$

$$0<|r|<1$$

ex: Evaluate.

b) $\int_1^\infty \frac{1}{x} dx$ $= \lim_{b\to\infty} \int_1^b \frac{1}{x} dx = \lim_{b\to\infty} \ln|x| \Big|_1^b$

$= \lim_{b\to\infty} \left(\ln b - \underset{0}{\cancel{\ln 1}} \right) = \infty$

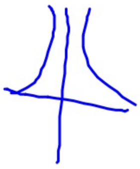
divergent

- A helpful theorem...

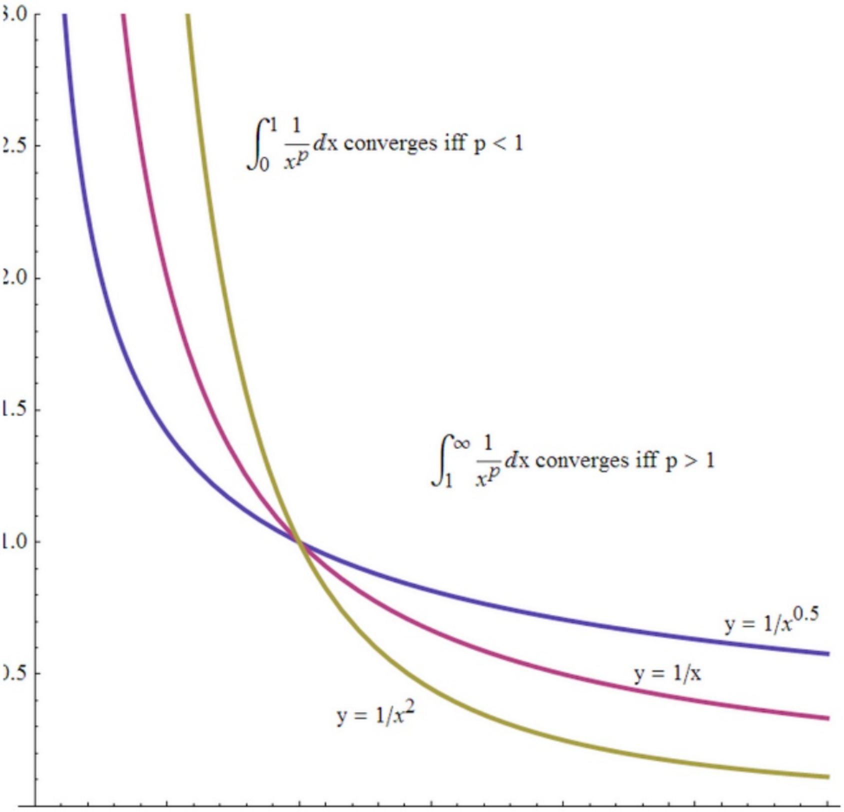
$$\boxed{\int_a^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{convergent,} & p>1 \\ \text{divergent,} & p\le 1 \end{cases}}$$

$$a>0 \qquad p\in R$$

$$\frac{1}{x^2}$$



A helpful graph



ex: Evaluate.

c) $\int_0^\infty \frac{1}{e^x} dx = \lim_{b \to \infty} \int_0^b e^{-x} dx = \lim_{b \to \infty} \left(-e^{-x} \right) \Big|_0^b$

$$\lim_{b \to \infty} \left(-\cancel{\frac{1}{e^b}}^{\nearrow 0} + 1 \right)$$

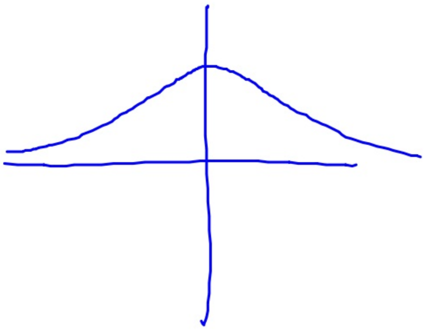
$$= 1$$

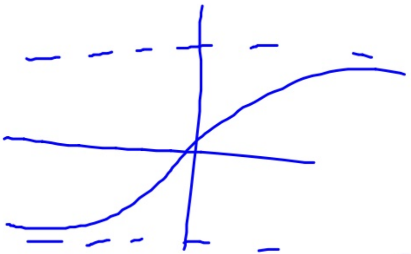
ex: Evaluate.

d) $\int_0^{\infty} \frac{1}{x^2+1} dx$

$$\lim_{b \to \infty} \int_0^b \frac{1}{x^2+1} dx$$

$$\boxed{\frac{\pi}{2}}$$





ex: Evaluate.

e) $\int_1^\infty (1-x)e^{-x}\,dx = \lim_{b\to\infty} \int_1^b \left(e^{-x} - xe^{-x}\right)dx$

$u = x \qquad dv = e^{-x}$

$du = 1 \qquad v = -e^{-x}$

$0 \qquad e^{-x}$

$$= \lim_{b\to\infty} \left(\cancel{-e^{-x}}\right)\Big|_1^b - \left(-xe^{-x} \cancel{-e^{-x}}\right)\Big|_1^b$$

$$\lim_{b\to\infty} xe^{-x}\Big|_1^b = \lim_{b\to\infty}\left(\overset{0}{\underset{\frac{b}{e^b}}{\cancel{be}^{-b}}} - e^{-1}\right) = -\frac{1}{e}$$

ex: Evaluate.

Pick an arbitrary point to split up the integral.

f) $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$

$$\lim_{a\to-\infty} \int_a^0 \frac{e^x}{1+e^{2x}} dx + \lim_{b\to\infty} \int_0^b \frac{e^x}{1+e^{2x}} dx$$

$$\lim_{a\to-\infty} \arctan e^x \Big|_a^0 + \lim_{b\to\infty} \arctan e^x \Big|_0^b$$

$$\frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$$

ex: Evaluate.

g) $\int_0^1 \frac{1}{\sqrt[3]{x}} dx = \lim_{a \to 0^+} \int_a^1 x^{-1/3} dx$

$$= \lim_{a \to 0^+} \frac{3}{2} x^{2/3} \Big|_a^1 = \lim_{a \to 0^+} \left(\frac{3}{2} - \overset{0}{\cancel{\frac{3}{2} a^{2/3}}} \right)$$

$$\frac{3}{2}$$

ex: Evaluate.

h) $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{b \to 1^-} \arcsin x \Big|_0^b$

$= \lim_{b \to 1^-} \left(\underset{\frac{\pi}{2}}{\arcsin b} - \underset{0}{\cancel{\arcsin 0}} \right)$

ex: Evaluate.

i) $\int_{1}^{3} \frac{5}{(x-2)^{8/3}}dx = \lim_{a\to 2^-}\int_{1}^{a} \frac{5}{(x-2)^{8/3}}dx + \lim_{b\to 2^+}\int_{b}^{3} \frac{5}{(x-2)^{8/3}}dx$

$$\lim_{a\to 2^-}\left(-3(x-2)^{-5/3}\right)\Big|_{1}^{a} + \lim_{b\to 2^+}\left(-3(x-2)^{-5/3}\right)\Big|_{b}^{3}$$

$$\lim_{a\to 2^-}\left(\overset{\infty}{\boxed{\frac{-3}{(\underset{1.99}{a}-2)^{5/3}}}} + \frac{3}{-1}\right)$$

Diverges

ex:

a) Find the area of the unbounded region lying between the graph of $f(x)=\frac{1}{x}$ and the x-axis ($x \geq 1$).

$\int_1^\infty \frac{1}{x}dx$ Diverges

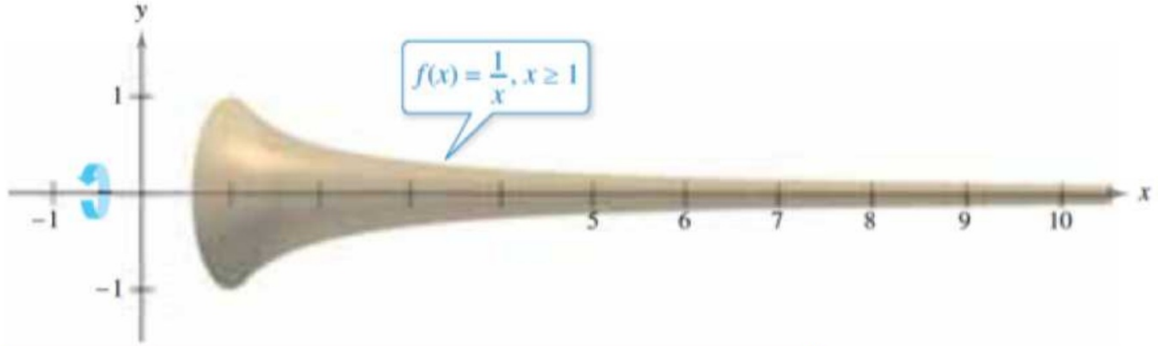
ex:

b) Find the volume of the solid formed by revolving the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the x-axis ($x \geq 1$).

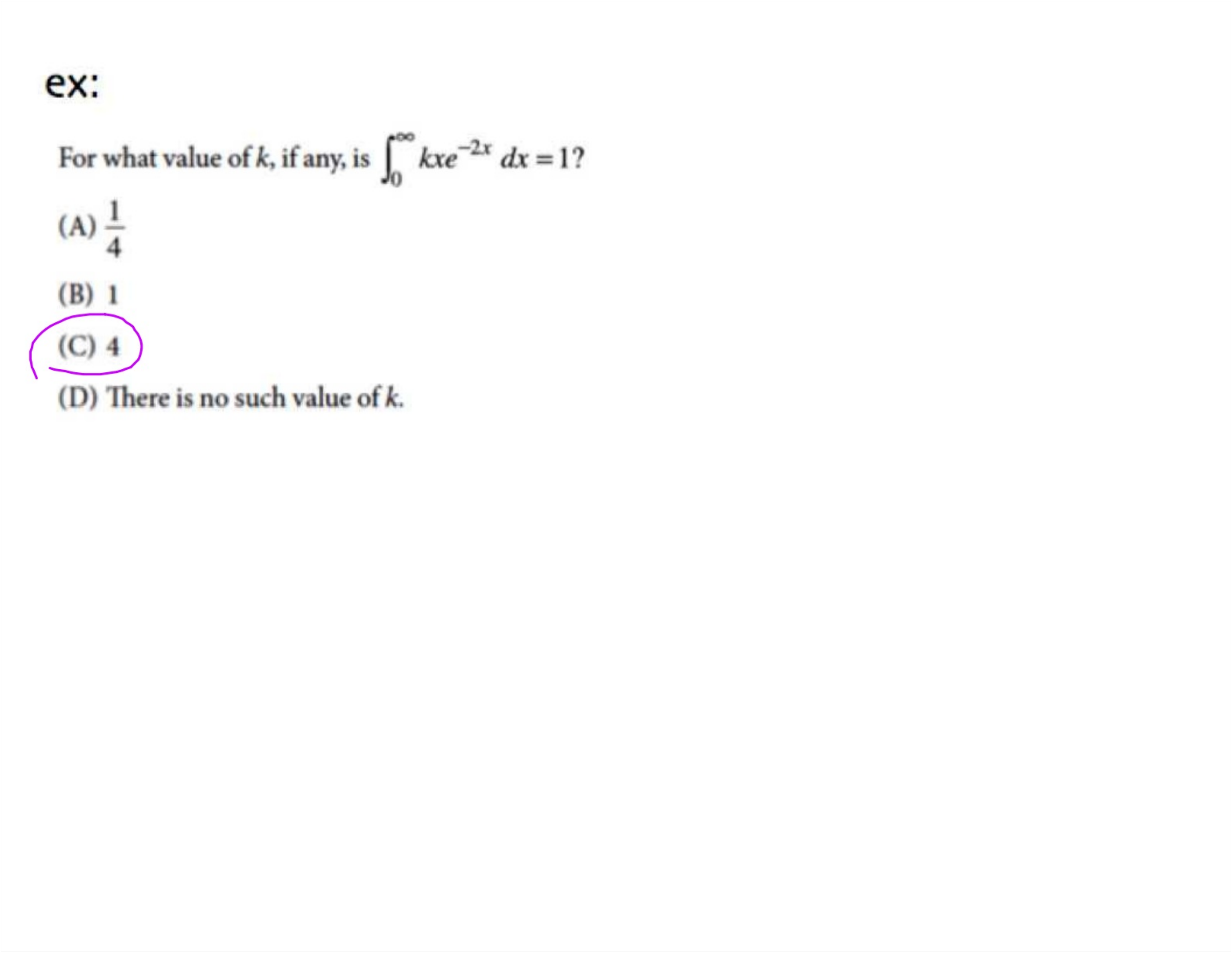
$$\left(\frac{1}{x} - 0\right)^2$$

$$\pi \int_1^\infty \frac{1}{x^2}\,dx = \pi$$

FUN FACT:



Gabriel's Horn has a finite volume and an infinite surface area.



ex:

For what value of k, if any, is $\int_0^\infty kxe^{-2x}\,dx = 1$?

(A) $\frac{1}{4}$

(B) 1

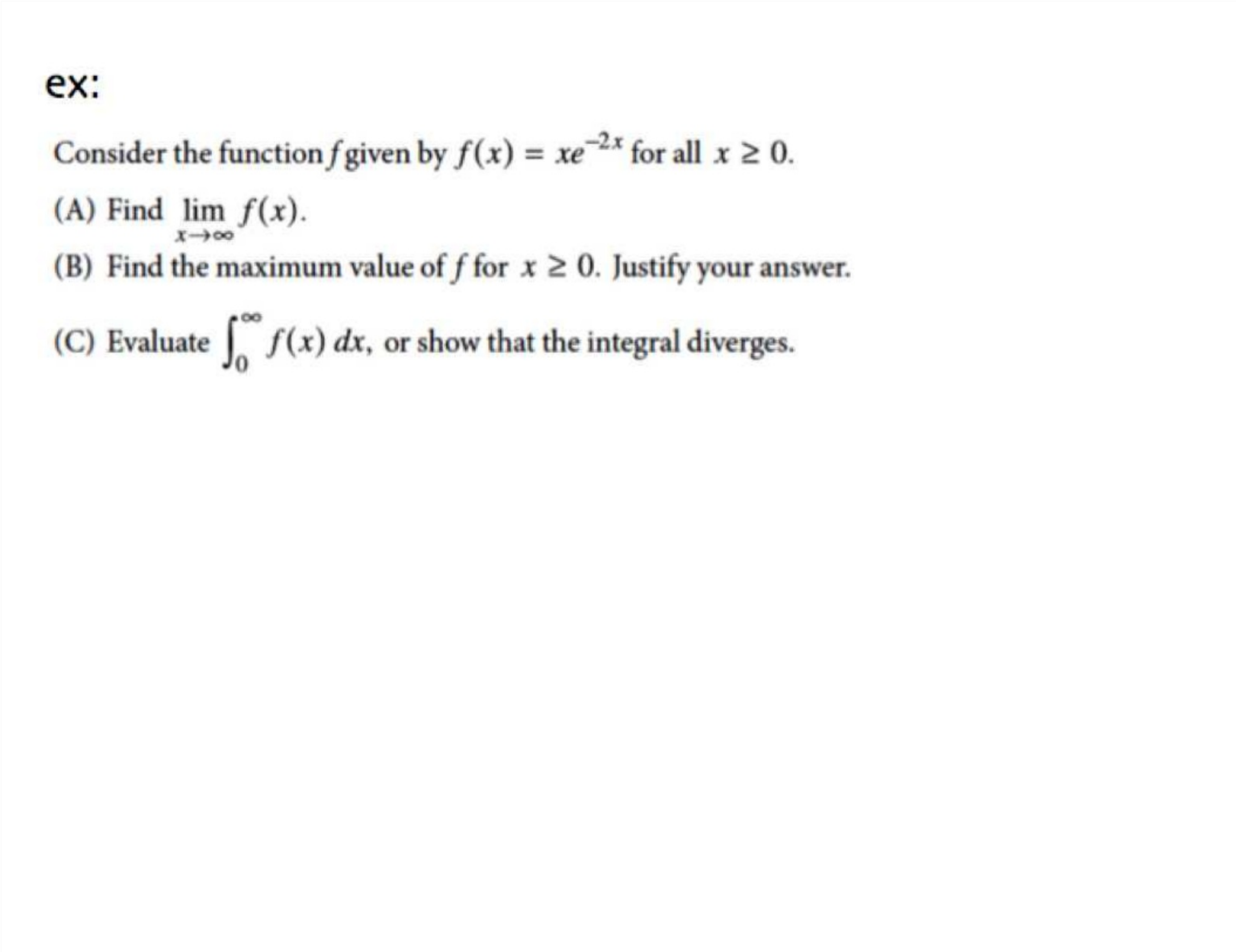
[✓ (C) 4]

(D) There is no such value of k.

ex:

$\int_1^\infty \frac{1}{x^p}\,dx$ and $\int_0^1 \frac{1}{x^p}\,dx$ both diverge when $p =$

(A) 2 [✓ (B) 1] (C) $\frac{1}{2}$ (D) 0 (E) -1



ex:

Consider the function *f* given by $f(x) = xe^{-2x}$ for all $x \geq 0$.

(A) Find $\lim_{x \to \infty} f(x)$.

(B) Find the maximum value of f for $x \geq 0$. Justify your answer.

(C) Evaluate $\int_0^{\infty} f(x)\,dx$, or show that the integral diverges.

16.) $\lim_{x\to\infty} x\ln\left(\frac{x+1}{x-1}\right)$ $\qquad \infty\cdot 0$

$$\lim_{x\to\infty} \frac{\ln(x+1)-\ln(x-1)}{\frac{1}{x}} \overset{L'Hop}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}-\frac{1}{x-1}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x-1-(x+1)}{(x+1)(x-1)} \cdot \frac{-x^2}{1} = \lim_{x\to\infty} \frac{2x^2}{(x+1)(x-1)} = 2$$

23.) $\lim_{x \to 0^+} \left(\frac{10}{x} - \frac{3}{x^2} \right)$

$\lim_{x \to 0^+} \left(\frac{10x - 3}{x^2} \right)$

$\frac{10(.01) - 3}{(.01)^2} = \frac{(-)}{\text{small}}$

$-\infty$

19.) $\lim\limits_{x \to \infty} \frac{x^2}{e^{-x}}$

$$\frac{100^2}{e^{-100}} = 100^2 \cdot e^{100} = \infty$$

∞

26.) $\lim\limits_{x \to \infty} x \sin\frac{1}{x} = \lim\limits_{x \to \infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}$

$$\overset{\text{L'HOP}}{=} \lim_{x \to \infty} \frac{\left(\cos\frac{1}{x}\right) \cancel{-\frac{1}{x^2}}}{\cancel{-\frac{1}{x^2}}} = 1$$

27.) $\lim_{x \to 0^+} x^{1/x} = 0$

$(.01)^{1/.01}$

$(.01)^{\text{Big}}$

28.) $\lim\limits_{x\to\infty} x^{1/x}$ $\qquad \infty^0$

$\ln y = \lim\limits_{x\to\infty} \ln x^{1/x}$

$\ln y = \lim\limits_{x\to\infty} \frac{\ln x}{x} \overset{\text{L'HOP}}{=} \lim\limits_{x\to\infty} \frac{1}{x} = 0$

$e^{\ln y} = e^{0}$

$y = 1$