7.7 Indeterminate Forms and L'Hopital's Rule

- 7 Indeterminate Forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^{\infty}, \quad 0^{0}, \quad \infty^{0}, \quad \infty \cdot 0, \quad \infty - \infty$$

NOT INDETERMINATE FORMS:

$$\infty \cdot \infty$$

$$\infty + \infty$$

L'Hopital's Rule

THEOREM 7.4 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

ex: Find the limit or explain why it does not exist.

a) $\lim_{x\to 0} \dfrac{e^{2x}-1}{x}$ $\overset{\text{L'Hop}}{=}$ $\lim_{x\to 0} \dfrac{2e^{2x}}{1} = 2$

$\boxed{\frac{0}{0}}$

ex: Find the limit or explain why it does not exist.

$\frac{\infty}{\infty}$

b) $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

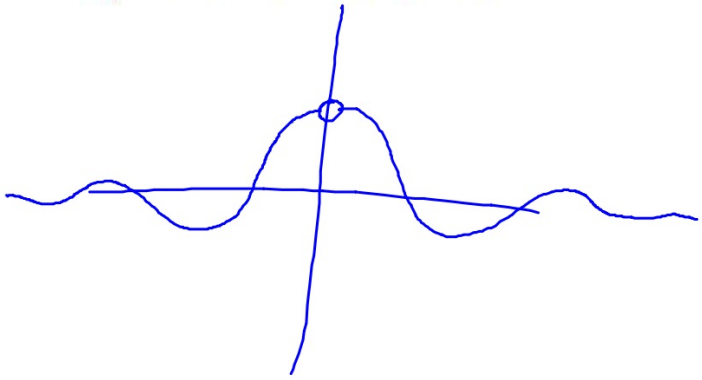
ex: Find the limit or explain why it does not exist.

c) $\lim_{x\to-\infty} e^x x^2 = \lim_{x\to-\infty} \frac{x^2}{e^{-x}}$ $\quad 0\cdot\infty \to \frac{\infty}{\infty}$

$\overset{L'Hop}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \overset{L'Hop}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$

ex: Find the limit or explain why it does not exist.

d) $\lim_{x\to\infty} \frac{\sin x}{x}$ = D



$\frac{\pm 1}{\infty}$

ex: Find the limit or explain why it does not exist.

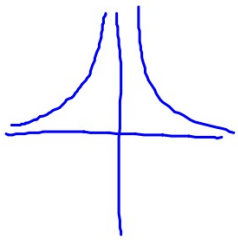
e) $\lim_{x \to \infty} \sin x$ dne (oscillating)

ex: Find the limit or explain why it does not exist.

f) $\lim_{x \to 5} \frac{x+3}{x-\sqrt{7}} = \frac{8}{5-\sqrt{7}}$

ex: Find the limit or explain why it does not exist.

g) $\lim_{x\to 6} \frac{x+2}{(x-6)^2}$ ∞

$\frac{1}{x^2}$ 

ex: Find the limit or explain why it does not exist.

h) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ $\quad 1^\infty$

Take the ln of both sides

$\ln y = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x$

$\ln y = \lim_{x\to\infty} \left(x \ln\left(1+\frac{1}{x}\right)\right)$

$\ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$\ln y \overset{\text{LHop}}{=} \lim_{x\to\infty} \frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$

$\ln y = 1$

$y = e$

$\ln\left(1+\frac{1}{x}\right) = \ln\frac{x+1}{x}$

$\ln(x+1) - \ln x$

$\frac{1}{x+1} - \frac{1}{x}$

MEMORIZE:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$$

more generally:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}$$

ex: Find the limit or explain why it does not exist.

i) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-15x} = e^{-15}$

j) $\lim_{x\to\infty}\ln\left(1+\frac{7}{x}\right)^{4x} = \ln e^{28} = 28$

ex: Find the limit or explain why it does not exist.

k) $\lim_{x \to 0^+} (\sin x)^x$

$$\ln y = \lim_{x \to 0^+} \ln(\sin x)^x$$

$$\ln y = \lim_{x \to 0^+} \left(x \cdot \ln(\sin x) \right)$$

$$\ln y = \lim_{x \to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}}$$

0

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ex: Find the limit or explain why it does not exist.

b) $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

$$\lim_{x\to 1+}\left(\frac{x-1-\ln x}{\ln x\,(x-1)}\right) \overset{L'HOP}{=} \lim_{x\to 1+}\left(\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}\right)$$

$$= \lim_{x\to 1+}\left(\frac{x-1}{x\ln x+x-1}\right) \overset{L'HOP}{=} \lim_{x\to 1+}\left(\frac{1}{2+\ln x}\right) = \frac{1}{2}$$

ex: Find the limit or explain why it does not exist.

m) $\lim_{x\to\infty}\left(\frac{2x-1}{4x+3}\right)\left(1-\frac{17}{x}\right)^{x}\arctan x$

$\left(\frac{1}{2}\right)\left(e^{-17}\right)\left(\frac{\pi}{2}\right)$

$\frac{\pi e^{-17}}{4}$

ex:

$$\lim_{x \to 0} \frac{7x - \sin x}{x^2 + \sin(3x)} =$$

(A) 6

(B) 2

(C) 1

(D) 0

(3) $\lim_{x \to 0^-} \frac{e^x}{3x}$

$\frac{1}{3(-.0001)}$

$-\infty$

(11) $\lim_{x \to \infty} 3x \ln\left(1+\frac{1}{x}\right)$

$\infty \cdot 0$

$3 \lim_{x \to \infty} x \ln\left(1+\frac{1}{x}\right)$

$3 \lim_{x \to \infty} \ln\left(1+\frac{1}{x}\right)^x$

3

(9) $\lim_{x \to 0} \frac{e^{3x} - 2^x}{3^x}$

0

29.) $\lim_{x \to \infty} \frac{x}{\sqrt{x^2+1}}$

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