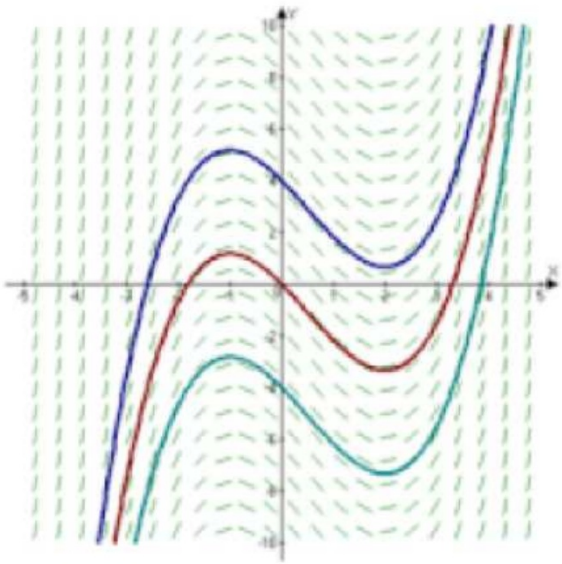
5.1/5.3 Slope Fields & Differential Equations

What is a Slope Field?

A slope field is a set of slopes that are possible for a differential equation

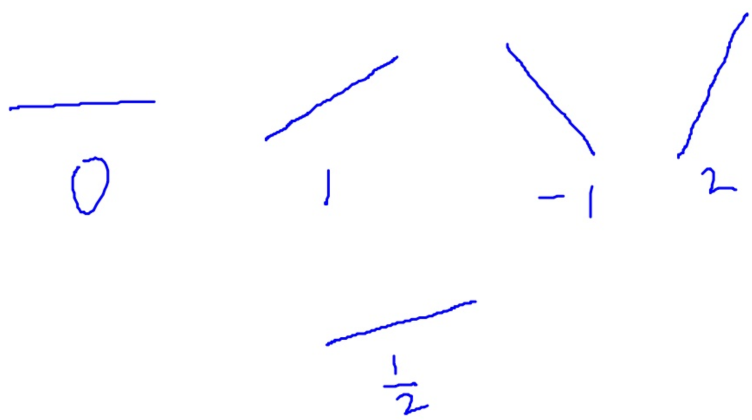


*See printout.

Sketching Slope Fields

1. *Calculate the slopes at the values indicated on the coordinate plane*

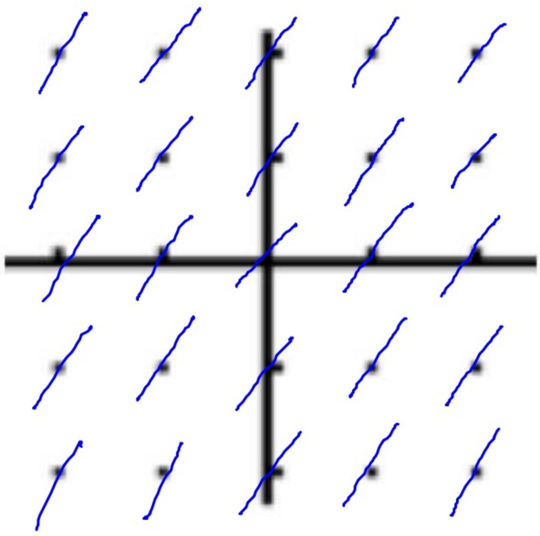
2. *Draw in 'segment' slopes*



Ex 1: Sketch each slope field.

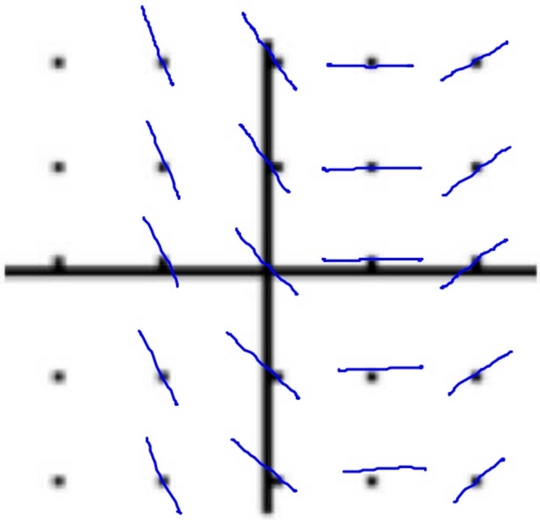
a) $\frac{dy}{dx} = 2$

$y = 2x + c$



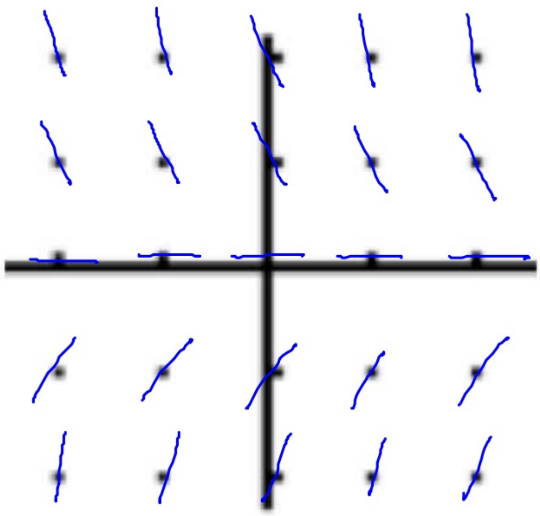
Ex 1: Sketch each slope field.

b) $\dfrac{dy}{dx} = x - 1$



Ex 1: Sketch each slope field.

c) $\dfrac{dy}{dx} = -3y$



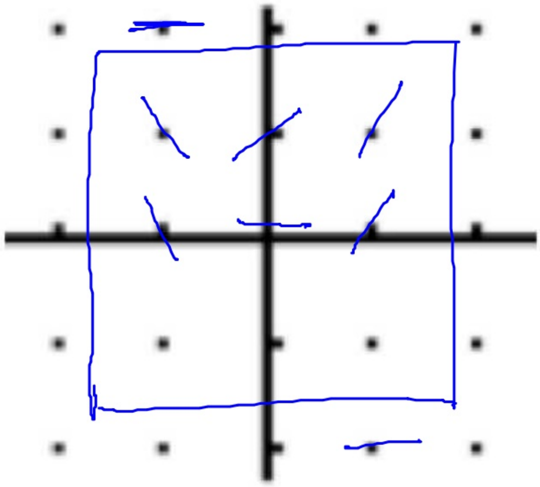
Ex 1: Sketch each slope field.

d) $\frac{dy}{dx} = 2x + y$

(0,1) : 1

(1,0) : 2

(1,1) : 3

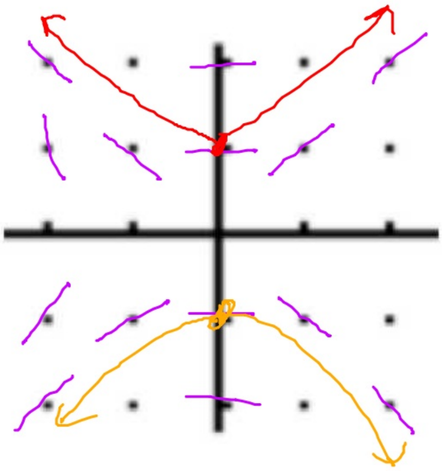


Ex 1: Sketch each slope field.

e) $\dfrac{dy}{dx} = y + xy$

Ex 2: Consider the differential equaiton given by $\frac{dy}{dx} = \frac{x}{y}$.

a) Sketch a slope field.



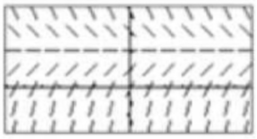
b) Sketch the solution curve that passes through the point (0, 1).

c) Sketch the solution curve that passes through the point (0, -1).

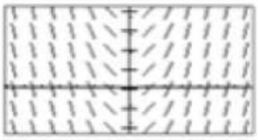
Matching Slope Fields with Differential Equations

Ex 3: Match each differential equation with the slope field.

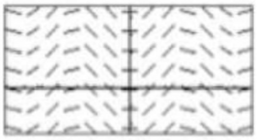
(A)



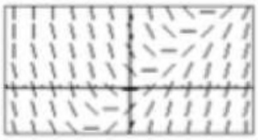
(B)



(C)



(D)



I. $\frac{dy}{dx} = \sin x$ C

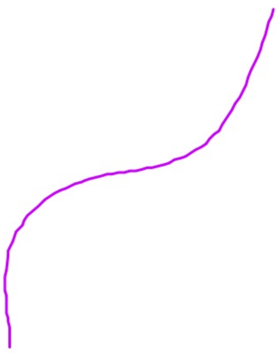
II. $\frac{dy}{dx} = x - y$ D

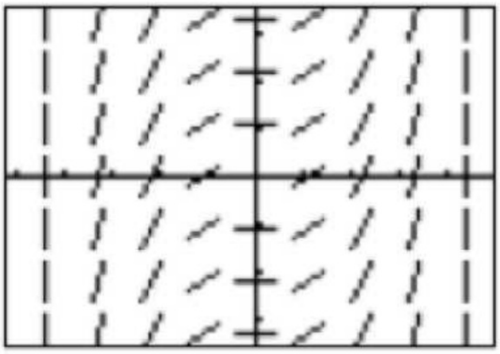
III. $\frac{dy}{dx} = 2 - y$ A

IV. $\frac{dy}{dx} = x$ B

Matching Slope Fields with Equations

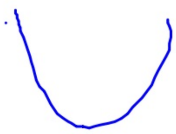
Ex 4: The slope field for a certain differential equation is shown below. Which of the following could be a particular solution to the differential equation?

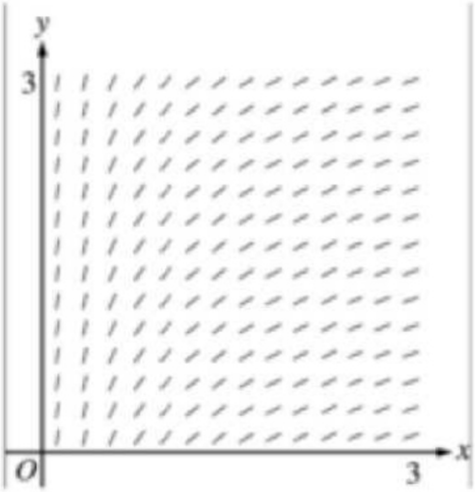




(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = x^2$ (d) $y = \frac{1}{6}x^3$ (e) $y = \ln x$

Ex 5: The slope field for a certain differential equation is shown below. Which of the following could be a particular solution to the differential equation?





(a) $y = x^2$ (b) $y = e^x$ (c) $y = e^{-x}$ (d) $y = \cos x$ (e) $y = \ln x$

Ex 5: Verify the solution of the differential equation.

a)

Solution	Differential Equation
$y = e^{-2x}$	$3y' + 5y = -e^{-2x}$

$y' = -2e^{-2x}$

$3(-2e^{-2x}) + 5e^{-2x} = -e^{-2x}$

$-6e^{-2x} + 5e^{-2x} = -e^{-2x}$ ✓

Ex 5: Verify the solution of the differential equation.

b)

Solution	Differential Equation
$y = 3\cos x + \sin x$	$y'' + y = 0$

$y' = -3\sin x + \cos x$

$y'' = -3\cos x - \sin x$

$-3\cos x - \sin x + 3\cos x + \sin x = 0$

$0 = 0$ ✓

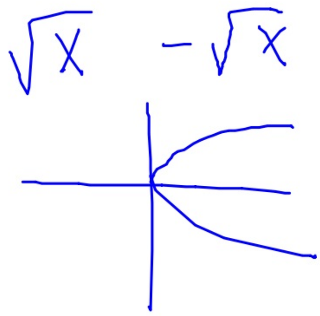
Two Types of Solutions to Differential Equations

1. *General solutions (+C)*

2. *Particular solutions, where you need an initial condition*

Ex 7: Find the general solution.

a) $y' = \dfrac{2x}{y}$



$$\frac{dy}{dx} = \frac{2x}{\cancel{y}}$$

Separate variables (y's with the dy AND x's with the dx)

$$\int y\,dy = \int 2x\,dx$$

$$\frac{1}{2}y^2 + C_1 = x^2 + C_2$$

$$\frac{1}{2}y^2 = x^2 + C$$

$$y^2 = 2x^2 + C$$

$$y = \pm\sqrt{2x^2 + C}$$

Ex 7: Find the general solution.

b) $y' = 3y$

leave the constant on the right; keep the 'y' side simple

$$\frac{dy}{dx} = 3y$$

$$\frac{dy}{y} = 3\,dx$$

$$e^{\ln|y|} = e^{3x + C}$$

$$|y| = Ce^{3x}$$

$$e^{3x+C}$$

$$e^{3x} \cdot e^{C} \to C$$

$$Ce^{3x}$$

Ex 8: Find the particular solution.

a) $y' = 7y$, (10, 1)

When you find C, plug in C to that specific equation

$$\frac{dy}{y} = 7\,dx$$

$$\ln|y| = 7x + C$$

$$\ln 1 = 70 + C$$

$$-70 = C$$

$$\ln|y| = 7x - 70$$

$$e^{\ln|y|} = e^{7x-70}$$

$$\boxed{y = e^{7x-70} \quad \text{or} \quad y = \frac{e^{7x}}{e^{70}}}$$

Ex 8: Find the particular solution.

b) $y' = \dfrac{x}{y}$ (0, -1)

$y\,dy = x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$

$y^2 = x^2 + C$

$1 = C$

$y^2 = x^2 + 1$

$y = \pm\sqrt{x^2+1}$

$y = -\sqrt{x^2+1}$

Ex 8: Find the particular solution.

c) $y' = \dfrac{y}{x^2}$, (1, 3)

$$\frac{dy}{y} = \frac{1}{x^2}\,dx$$

$$e^{\ln|y|} = e^{-\frac{1}{x}+C}$$

$$|y| = e^{-\frac{1}{x}+C} \quad e^{-\frac{1}{x}} \cdot e^{C}$$

$$|y| = Ce^{-\frac{1}{x}}$$

$$3 = Ce^{-1}$$

$$3e = C$$

$$y = 3e^{1} \cdot e^{-\frac{1}{x}} \text{ or}$$

$$y = 3e^{1-\frac{1}{x}}$$

Ex 8: Find the particular solution.

d) $y\sqrt{1-x^2}\,y' - x\sqrt{1-y^2} = 0$, (0, 1)

$$y\sqrt{1-x^2}\,\frac{dy}{dx} = x\sqrt{1-y^2}$$

$$\frac{y}{\sqrt{1-y^2}}\,dy = \frac{x}{\sqrt{1-x^2}}\,dx$$

Ex 9: The rate of change of y with respect to x is proportional to the difference between x and 4. Write a differential equation.

"K"

$$\frac{dy}{dx} = K(x-4)$$

Directly: multiply
Inversely : divide

$$dy = K(x-4)\,dx$$

$$y = K\left(\frac{x^2}{2} - 4x + c\right)$$

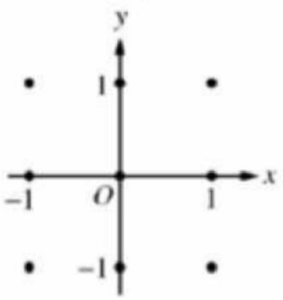
Ex 10: The rate of change of y with respect to x varies directly with the square of y. Write a differential equation.

$$\frac{dy}{dx} = ky^2$$

Ex 11:

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

Ex 12:

Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\dfrac{3x^2+1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

(c) Find $f(x)$ by solving the separable differential equation $\dfrac{dy}{dx} = \dfrac{3x^2+1}{2y}$ with the initial condition $f(1) = 4$.

(d) Use your solution from part (c) to find $f(1.2)$.

Ex 13:

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

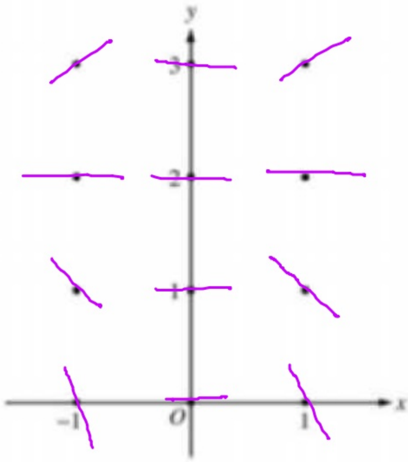
Ex 14:

Consider the differential equation $\frac{dy}{dx} = x^4(y-2)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated. **(Note: Use the axes provided in the test booklet.)**

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are negative.

(c) Find the particular solution $\boxed{y = f(x)}$ to the given differential equation with the initial condition $f(0) = 0$.



b.) all points below $y=2$, but $x \neq 0$

c.) $\frac{dy}{y-2} = x^4 \, dx$

$\ln|y-2| = \frac{x^5}{5} + C$

$e^{\ln|y-2|} = e^{x^5/5}$

$|y-2| = Ce^{x^5/5}$

$2 = C$

$$\boxed{y = -2e^{x^5/5} + 2}$$

$$\ln|y-2| = \frac{x^5}{5} + C$$

$$\ln|2| = C$$

$$\ln|2| = C$$

$$e^{\ln|y-2|} = e^{\frac{x^5}{5} + \ln 2}$$

$$|y-2| = 2e^{x^5/5} \Rightarrow -y+2 = 2e^{x^5/5}$$

$$+y = -2e^{x^5/5} + 2$$