2.6 Derivatives of Inverse Functions

Review: Two functions are inverse functions if

$f(x) = x^3$

$x = y^3$

$\sqrt[3]{x} = y$

$f^{-1}(x) = \sqrt[3]{x}$

$$\boxed{\begin{gathered}\left(f \circ f^{-1}\right)(x) = x \\ \text{and} \\ \left(f^{-1} \circ f\right)(x) = x\end{gathered}}$$

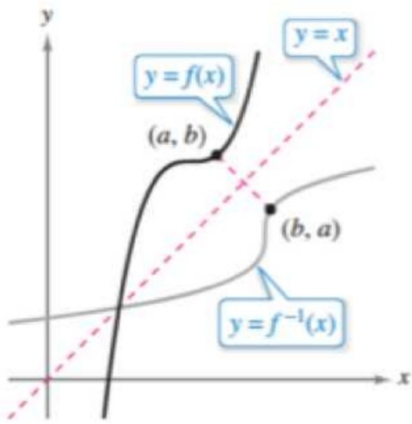
$(\sqrt[3]{x})^3 = x$

$(x^3)^{1/3} = x$

Inverse Properties

- The graphs of f and f^{-1} are reflections about the line y=x.
- The domain of f is the range of f^{-1}.
- The range of f is the domain of f^{-1}.
- A function has an inverse if and only if it is one-to-one.

(monotonic)
·always increasing
or
·always decreasing

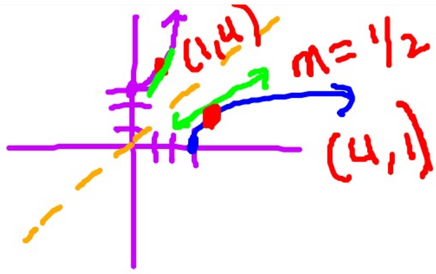


Derivative of an inverse function

$f(x) = \sqrt{x-3}$

D: $[3, \infty)$

R: $[0, \infty)$



$f'(4)$?

$f'(x) = \dfrac{1}{2\sqrt{x-3}}$

$\boxed{f'(4) = \frac{1}{2}}$

$f^{-1}(x) = x^2 + 3, \; x \geq 0$

$\boxed{(f^{-1})'(1) = 2}$

The slope of a function and its inverse function have a reciprocal relationship.

$$f(x) = x^5 + 2x - 1$$

$$2 = x^5 + 2x - 1$$

$$x = 1$$

$$f'(x) = 5x^4 + 2$$

$$f'(1) = 7$$

reciprocal

$$(f^{-1})'(2) = \frac{1}{7}$$

$$f : (1, 2)$$

$$f^{-1} : (2, 1)$$

ex: If $f(x) = x^5 + 2x^3 + x - 1$ and

$(f \circ g)(x) = (g \circ f) = x$ find $g'(3) = \frac{1}{12}$

$f: (1, 3)$

$g: (3, 1)$

$3 = x^5 + 2x^3 + x - 1$

$1 = x$

$f'(x) = 5x^4 + 6x^2 + 1$

$f'(1) = 12$

reciprocal

3. 18% answered correctly

If $f(x) = x^3 + x$ and $h(x)$ is the inverse of $f(x)$, then $h'(2)$ is

A) $\frac{1}{13}$ B) $\frac{1}{4}$ C) 1 D) 4 E) 13

$h(2, \ )$

$f(\ , 2)$

$2 = x^3 + x$

$1 = x$

$f'(x) = 3x^2 + 1$

$f'(1) = 4$

4. 14% answered correctly

Let f be a differentiable function such that $\boxed{f(3) = 15}$, $\boxed{f'(3) = -8}$ and $\boxed{f'(6) = -2}$ $\boxed{f(6) = 3}$

The function g is differentiable and $g(x) = f^{-1}(x)$ for all x. What is the value of $g'(3)$?

A) $-\frac{1}{2}$ B) $-\frac{1}{8}$ C) $\frac{1}{6}$ D) $\frac{1}{3}$ E) The value of $g'(3)$ cannot be determined from the information given.

$g: (3, \ )$

$f(\underline{6}, 3)$

6.

Suppose f is a one-to-one function, which is differentiable for all real numbers x. The table below gives some of the values of $f(x)$ and $f'(x)$:

x	$f(x)$	$f'(x)$
1	2	$\frac{7}{6}$
2	3	$\frac{7}{6}$
3	5	$\frac{19}{6}$
4	10	$\frac{43}{6}$

(a) Write an equation of the tangent line, T_1, to the function $f(x)$ at $x = 3$.

(b) Write an equation of the normal line, N_1, to the function $f(x)$ at $x = 3$.

(c) Write an equation of the tangent line, T_2, to the function $f^{-1}(x)$ at $x = 3$.

$y - 2 = \frac{6}{7}(x - 3)$

$f^{-1}: (3, 2)$

$f: (2, 3)$

$f'(2) = \frac{7}{6}$

reciprocal

10. Mean Score 0.95 (2007)

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table gives values of the functions and their first derivatives at selected values of x. The function h is given by $\boxed{h(x) = f(g(x)) - 6.}$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

$h(1) = 3$

$h(3) = -7$

$h(3) < -5 < h(1)$

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)dt$. Find the value $w'(3)$.

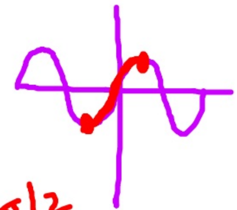
(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

$y - 1 = \frac{1}{5}(x - 2)$

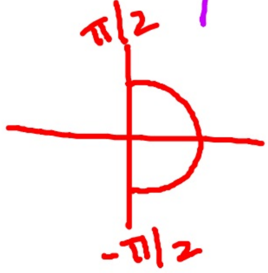
$g^{-1}(2, 1)$

$g(1, 2)$

2.6 Derivatives of Inverse Functions Cont.

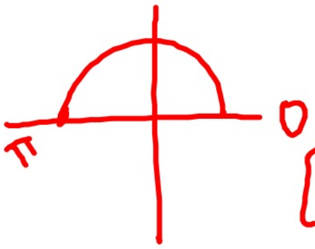


ex: Evaluate.



a) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



b) $\arccos(-1) = \pi$

$[0, \pi]$

Belly functions

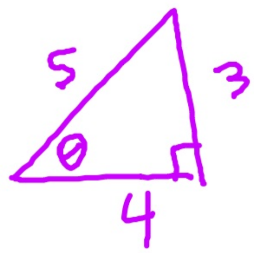
arcsin x
arctan x
arccsc x

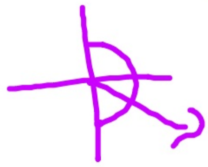
Sunset functions

arccos x
arccot x
arcsec x

ex: Evaluate.

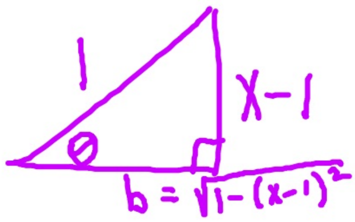
c) $\sin\left(\tan^{-1}\left(-\frac{3}{4}\right)\right) = -\frac{3}{5}$





d) $\cos\left(\arcsin\left(\frac{x-1}{1}\right)\right)$

$= \sqrt{1-(x-1)^2}$



$1^2 = (x-1)^2 + b^2$

$1-(x-1)^2 = b^2$

Derivatives of Inverse Trigonometric Functions

THEOREM 2.18 **Derivatives of Inverse Trigonometric Functions**

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

We will re-visit the proofs of these derivatives next week :)

ex: Find the derivative.

a) $y = \sin^{-1}(2x)$

$$y' = \frac{2}{\sqrt{1-4x^2}}$$

find $y'\left(\frac{1}{4}\right)$

$$y'\left(\frac{1}{4}\right) = \frac{2}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\frac{\sqrt{3}}{2}} = \frac{4}{\sqrt{3}}$$

Write the equation of the tangent line at x = 1/4

$\left(\frac{1}{4}, \frac{\pi}{6}\right)$

$$y - \frac{\pi}{6} = \frac{4}{\sqrt{3}}\left(x - \frac{1}{4}\right)$$

p. 185

ex: Find the derivative.

b) $f(x) = \sec^{-1}\left(e^{7x}\right)$

$$f'(x) = \frac{7e^{7x}}{|e^{7x}|\sqrt{e^{14x}-1}} = \frac{7}{\sqrt{e^{14x}-1}}$$

ex: Find an equation of the tangent line to the graph of f at the given point.

$$y = \arctan\left(\frac{x}{2}\right), \quad x = -2 \qquad \left(-2, \frac{-\pi}{4}\right)$$

$$y' = \left(\frac{\frac{1}{2}}{1+\frac{x^2}{4}}\right)\frac{\cdot 4}{\cdot 4} = \frac{2}{4+x^2}$$

$$y'(-2) = \frac{1}{4}$$

$$\boxed{y + \frac{\pi}{4} = \frac{1}{4}(x+2)}$$