2.4 Chain Rule

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

THEOREM 2.11 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

$$y = (3x+2)^2 = 9x^2 + 12x + 4$$

$$y' = 2(3x+2)^1 \cdot 3 \qquad y' = 18x + 12$$

$$y' = 6(3x+2)$$

ex: Which function(s) are good candidates for the chain rule?

- $y = \sin\left(x^2 + 1\right)$ ✓
- $y = e^{2x}$ ✓
- $y = \ln\left(x - 5\right)$ ✓
- $f\left(x\right) = \sin x \cos x$
- $y = \dfrac{5}{x+1}$ $= 5(x+1)^{-1}$

ex: Find the derivative.

a) $y = \sin 2x$

$y' = \cos(2x) \cdot 2$

$y' = 2\cos 2x$

$\sin 2x + 1$

$\sin(2x+1)$

b) $y = \sin(\cos x)$

$y' = \cos(\cos x) \cdot -\sin x$

$y' = -\cos(\cos x)\sin x$

ex: Find the derivative.

c) $y = \sqrt{e^x + 1} = (e^x + 1)^{1/2}$

$y' = \frac{1}{2}(e^x + 1)^{-1/2} \cdot e^x$

$y' = \frac{e^x}{2\sqrt{e^x + 1}}$

d) $y = \cos^4 x = (\cos x)^4$

$y' = 4(\cos x)^3(-\sin x)$

$y' = -4\cos^3 x \sin x$

ex: Find the derivative.

e) $y = \tan^2 x = (\tan x)^2$

$y' = 2(\tan x)' \cdot \sec^2 x$

f) $y = \tan(x^2)$

$y' = \sec^2(x^2) \cdot 2x$

$y' = 2x\sec^2 x^2$

ex: Find the derivative.

g) $y = \sec(3x)$

$y' = \sec(3x)\tan(3x) \cdot 3$

ex: Find the derivative.

h) $y = \left(\dfrac{3x-1}{x^2+2}\right)^3$

$y' = 3\left(\dfrac{3x-1}{x^2+2}\right)^2\left(\dfrac{(x^2+2)3-(3x-1)2x}{(x^2+2)^2}\right)$ . . .

ex: Find the derivative.

i) $y = \dfrac{\sin^2 x}{4x-1}$

$$y' = \frac{(4x-1)2\sin x\cos x - \sin^2 x \cdot 4}{(4x-1)^2}$$

ex: Find the derivative.

j) $f(x) = x^2\sqrt{1-x^2}$

$(1-x^2)^{1/2}$

$f'(x) = x^2 \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) + (1-x^2)^{1/2} \cdot 2x$

$= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2}$ $\quad \frac{1}{2} - -\frac{1}{2}$

$= -x(1-x^2)^{-1/2}\left(x^2 - 2(1-x^2)^1\right)$

$= \dfrac{-x(3x^2-2)}{(1-x^2)^{1/2}}$

ex: Find the derivative.

k) $y = \cos^4(3x) = (\cos 3x)^4$

$y' = 4(\cos 3x)^3(-\sin 3x \cdot 3)$

$y' = -12\cos^3 3x \sin 3x$

$y = \cos 3x$

$y' = -3\sin 3x$

ex: $g(x) = \tan x, \qquad \frac{d^2y}{dx^2} = ?$

$g'(x) = \sec^2 x = (\sec x)^2$

$g''(x) = 2\sec x \cdot \sec x \tan x$

$= 2\sec^2 x \tan x$

ex: Write the equation of the tangent line to

$$y=\left(\frac{1}{x^2+9}\right)^2 \text{ at } x=1.$$

$y = (x^2+9)^{-2}$

$y' = -2(x^2+9)^{-3}\cdot 2x$

$= \frac{-4x}{(x^2+9)^3}$

$y'(1) = \frac{-1}{250}$

$\left(1, \frac{1}{100}\right)$

$y - \frac{1}{100} = \frac{-1}{250}(x-1)$

ex: Find the equation of the line that is tangent to $y = \sqrt{3x-1}$ and perpendicular to $3y + 2x = 3$.

ex:

$K'(0) = f'(g(0)) \cdot g'(0)$

$= f'(2) \cdot 5$

$= 3 \cdot 5 = 15$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	−1	2	5
1	−1	2	4	0
2	7	3	11	0.5

1. Based on the values in the table above.
If $K(x) = f(g(x))$, then $K'(0) =$

(A) 15 (B) 35 (C) −5 (D) −1 (E) 7

2. Let $g(x) = x \sin(x^2)$. Then $g'(x) =$

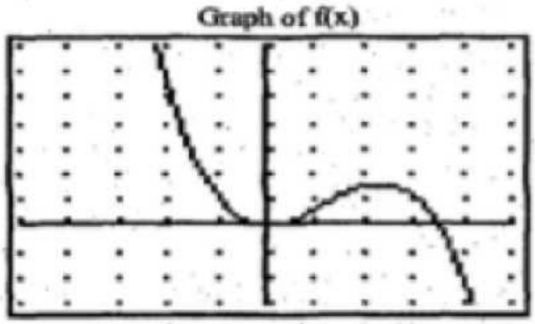
(A) $2x^2 \cos(x^2)$ (B) $-2x^2 \cos(x^2)$ (C) $2x^3 \cos(x^2)$

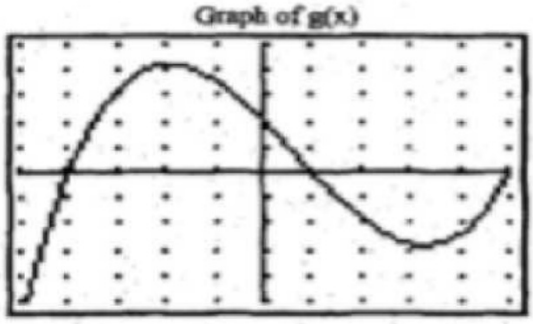
(D) $2x^2 \cos(x^2) + \sin(x^2)$ (E) $-2x^2 \cos(x^2) + \sin(x^2)$

$g'(x) = x \cdot \cos(x^2) 2x + \sin(x^2) \cdot 1$

ex:

Let f and g be the functions defined below





Let $h(x) = f(g(x))$ and let $k(x) = f(x^4)$

a) Evaluate h(-2), h(1), and h(2)

b) Is h'(-1) positive, negative, or equal to zero. Justify your answer.

c) Estimate the sign of h'(-2), h'(1), and h'(2).

ex: Find the derivative.

a) $y = \tan x \cot x$

$y = 1$

b) $y = \dfrac{2}{(x-3)^4} = 2(x-3)^{-4}$

c) $y = \left(\dfrac{x}{x^2+1}\right)^{-1}$

$y = \dfrac{x^2+1}{x} = x + x^{-1}$

d) $y = \dfrac{\tan x}{\sin x - 1}$

e) $y = 2\sin x \cos x$

$y = \sin 2x$

f) $y = \cot x \csc x$

$\dfrac{\cos x}{\sin^2 x}$

a) $y = \tan x \cot x$

b) $y = \dfrac{2}{(x-3)^4}$

c) $y = \left(\frac{x}{x^2 + 1} \right)^{-1}$

d) $y = \dfrac{\tan x}{\sin x - 1}$

e) $y = 2\sin x \cos x$

f) $y = \cot x \csc x$

ex: Find the indicated derivative at the given point using your calculator.

a) $\frac{d}{dx}(\ln x)\Big|_{x=8}$

$y' = \frac{1}{x}$ $y'(8) = \frac{1}{8}$

b) $f(x) = \tan^{-1} x, \quad f'(5)$

c) $y = \frac{3x+2}{\sqrt{x^2+5}}, \quad \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}}$

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Let f be the function given by $f(x)=\dfrac{x}{\sqrt{x^2-4}}$.

(a) Find the domain of f.

(b) Write an equation for each vertical asymptote to the graph of f.

(c) Write an equation for each horizontal asymptote to the graph of f.

(d) Find $f'(x)$.