2.3 Product & Quotient Rules

ex: Find the derivative.

a) $y=(x+1)(x^2-3)$ $= x^3 - 3x + x^2 - 3$

$y' = 3x^2 - 3 + 2x$

b) $g(x)=\dfrac{2x^2-1}{x}$ $= 2x - x^{-1}$

$g'(x) = 2 + x^{-2}$

c) $y = e^x \sin x$

d) $f(x) = \dfrac{x^2+1}{2x-1}$

THEOREM 2.8 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$y = (x+1)(x^2-3)$$

$$y' = (x+1)2x + (x^2-3)(1)$$

$$= 2x^2 + 2x + x^2 - 3$$

$$= 3x^2 + 2x - 3$$

ex: Which function(s) are good candidates for the product rule?

- $f(x) = (4)(\sin x)$
- ✓ $y = xe^{x}$
- ✓ $y = (x+1)(x^{2}-3)$
- $y = 2x^{3}$
- $g(x) = \sin(2x)$

ex: Find the derivative.

a) $y = xe^x$

$y' = x \cdot e^x + e^x \cdot 1 = xe^x + e^x = e^x(x+1)$

b) $g(x) = \sin 2x \;\; = 2\sin x \cos x$

$g'(x) = 2\left(\sin x(-\sin x) + \cos x \cdot \cos x\right)$

$= 2\left(\cos^2 x - \sin^2 x\right)$

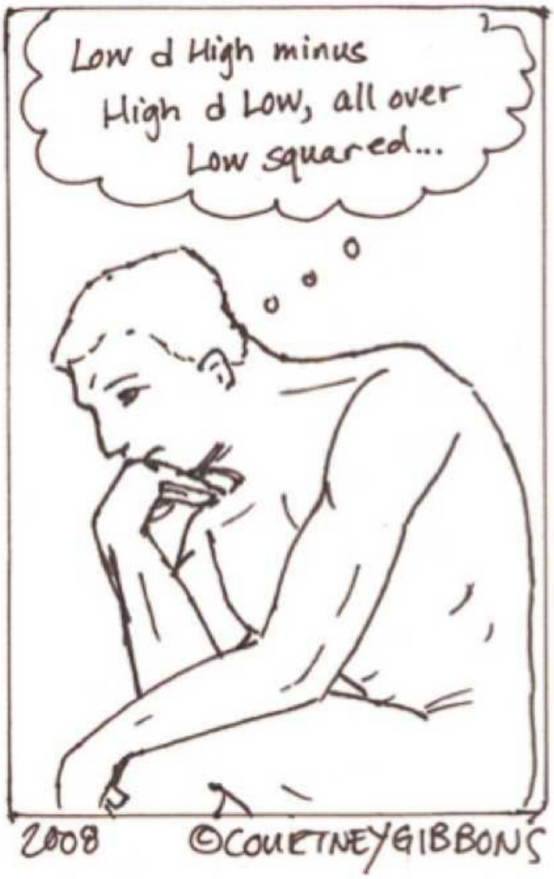
$= 2\cos 2x$

THEOREM 2.9 The Quotient Rule

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Remembering The Quotient Rule...



$$\frac{Lo \cdot dHi - Hi \cdot dLo}{Lo^2}$$

ex: Which function(s) are good candidates for the quotient rule?

- $g(x) = \frac{5}{x}$
- $y = \frac{x}{5}$ $= \frac{1}{5}x$
- ✓ $y = \frac{e^x}{x+1}$
- ✓ Yand ND $f(x) = \frac{x+1}{x}$
- ✓ $y = \tan x$

ex: Find the derivative.

a) $y = \dfrac{e^x}{x+1}$

$$y' = \frac{(x+1)e^x - e^x(1)}{(x+1)^2}$$

$$\boxed{y' = \frac{xe^x}{(x+1)^2}}$$

ex: Find the derivative.

b) $f(x) = x^3\left(1 - \frac{2}{x+1}\right) = x^3 - \frac{2x^3}{x+1}$

$$f'(x) = 3x^2 - \left(\frac{(x+1)6x^2 - 2x^3}{(x+1)^2}\right)$$

$$= \frac{3x^2}{1} - \left(\frac{4x^3+6x^2}{(x+1)^2}\right) = \frac{3x^2(x^2+2x+1) - (4x^3+6x^2)}{(x+1)^2}$$

$$= \frac{3x^4 + 2x^3 - 3x^2}{(x+1)^2}$$

ex: Find the derivative.

c) $f(x) = \dfrac{3 - \dfrac{1}{x+5}}{x-1}$

ex: Find the derivative.

c) $y = \tan x = \frac{\sin x}{\cos x}$

$$y' = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$y' = \sec^2 x$$

ex: Find the derivative.

d) $y = \cot x$

$y' = -\csc^2 x$

ex: Find the derivative.

e) $y = \sec x = \frac{1}{\cos x}$

$$y' = \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x \cdot 1}{\cos x \cos x}$$

$$y' = \tan x \sec x$$

ex: Find the derivative.

f) $y = \csc x$

$y' = -\cot x \csc x$

Trigonometric Derivatives

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

Remembering The Derivatives of Tangent, Cotangent, Secant and Cosecant...

*MEMORIZE THIS CHART

$\tan x$	$\sec x$	$\sec x$
$\cot x$	$\csc x$	$-\csc x$

ex: Find the equation of the tangent line to $y = x^2 \sec x$ at $x = \dfrac{\pi}{4}$.

$\left(\frac{\pi}{4}, \frac{\pi^2\sqrt{2}}{16}\right)$

$$y' = x^2 \cdot \sec x \tan x + \sec x \cdot 2x$$

$$y'\left(\frac{\pi}{4}\right) = \left(\frac{\pi^2}{16} \cdot \sqrt{2} + \frac{\sqrt{2}\,\pi}{2}\right)$$

$$y - \frac{\pi^2\sqrt{2}}{16} = \left(\frac{\pi^2}{16}\sqrt{2} + \frac{\sqrt{2}}{2}\pi\right)\left(x - \frac{\pi}{4}\right)$$

ex: Find the point(s), if any, at which $f(x) = \dfrac{x^2}{x^2+1}$ has a horizontal tangent.

0 slope

$\Rightarrow (0, 0)$

$f'(x) = \dfrac{2x}{(x^2+1)^2}$

set numerator = 0

$0 = 2x$

$0 = x$

ex:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	−1	2	5
1	−1	2	4	0
2	7	3	11	0.5

Based on the values in the table above,
if $\boxed{H(x) = ef(x) + \pi x,}$ then $\boxed{H'(0)}$ =

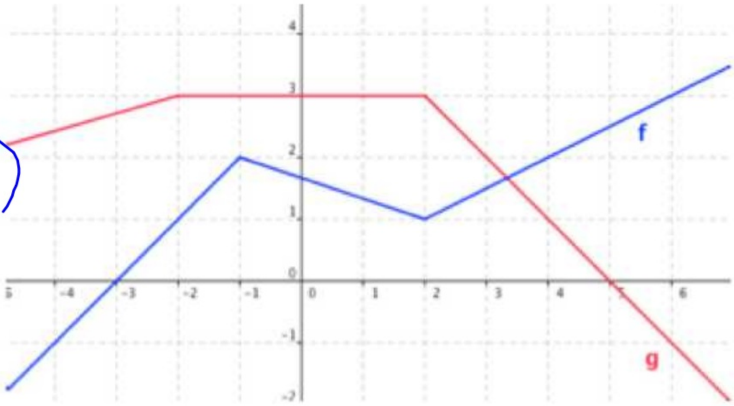
(A) $\pi - e$ (B) $e^x + \pi x$ (C) $e + \pi$ (D) e (E) $e^{-1} + \pi$

$H'(x) = ef'(x) + \pi$

$H'(0) = ef'(0) + \pi = -e + \pi$

ex: Let $p(x) = f(x)g(x)$ and $q(x) = \frac{f(x)}{g(x)}$.

$p(x) = f(x)g(x)$

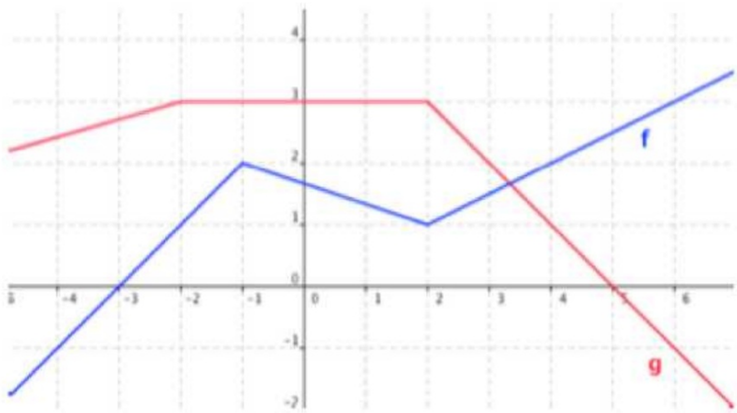


1. $p'(x) = f(4)g'(4) + f'(4)g(4)$

$p'(4) = (2)(-1) + \left(\frac{1}{2}\right)(1) = -2 + \frac{1}{2} = -1\frac{1}{2}$

*See printout.

$$q(x) = \frac{f(x)}{g(x)}$$



4. $q'(x) = \dfrac{g(-2)\,f'(-2) - f(-2)\,g'(-2)}{\left(g(-2)\right)^2}$

$q'(-2) = \cancel{(3)(1) - (1)(\text{und})}$ undefined

Higher Order Derivatives

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
$\vdots$					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

ex: Find the indicated derivative.

a) $f(x) = 3x^4 + 2, \qquad f'''(x) = ?$

$f'(x) = 12x^3$

$f''(x) = 36x^2$

$f'''(x) = 72x$

ex: Find the indicated derivative.

b) $y = \sin x$, $\qquad \dfrac{d^2 y}{dx^2} = ?$

$y' = \cos x$

$y'' = -\sin x$

ex: Find the indicated derivative.

c) $y = \sin x$, $\qquad \dfrac{d^5 y}{dx^5} = ?$

$y'' = -\sin x$

$y''' = -\cos x$

$y^{(4)} = \sin x$

$y^{(5)} = \cos x$

ex: Find the indicated derivative.

d) $y = \sin x$, $\qquad \dfrac{d^{205}y}{dx^{205}} = ?$

$$\begin{array}{r} 51 \\ 4\,\overline{)\,205} \\ \underline{20} \\ 5 \\ \underline{-4} \\ 1 \end{array}$$

$y^{(205)} = \cos x$

$y = \sin x$

$y' = \cos x$

$y'' = -\sin x$

$y''' = -\cos x$

$y^{(4)} = \sin x$

ex: Find the indicated derivative.

e) $g(x) = \cos x, \qquad g^{(163)}(x) = ?$

$$
\begin{array}{r}
40 \\
4\overline{)163} \\
16 \\
\hline
3 \\
0 \\
\hline
3
\end{array}
\quad R3
$$

$g^{(163)}(x) = \sin x$

$y = \cos x$

$y' = -\sin x$

$y'' = -\cos x$

$y''' = \sin x$

$y^{(4)} = \cos x$

ex: Find the indicated derivative.

f) $y = e^{x}, \qquad y^{(1111)} = ?$

FR 5

Let f be the function given by $f(x)=\frac{2x-5}{x^2-4}$.

a. Find the domain of f.

b. Write an equation for each vertical and each horizontal asymptote for the graph of f.

* c. Find $f'(x)$.

* d. Write an equation for the line tangent to the graph of f at the point (0,f(0)).

a.) $x \neq \pm 2$

b.) $x = 2, -2$ VA

$y = 0$ H.A

FR 18

Let f be the function that is given by $f(x) = \dfrac{ax+b}{x^2-c}$ and that has the following properties.

(i) The graph of f is symmetric with respect to the y-axis.

(ii) $\lim_{x \to 2^+} f(x) = +\infty$

(iii) $f'(1) = -2$

(a) Determine the values of a , b , and c .

(b) Write an equation for each vertical and each horizontal asymptote of the graph of f.

(c) Sketch the graph of f in the xy-plane provided below.