1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using dividing out and rationalizing techniques.
- Evaluate a limit using the Squeeze Theorem.

THEOREM 1.2 PROPERTIES OF LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$
2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Strategy for finding limits analytically

1. Direct Substitution
2. Algebraic techniques

(factoring or rationalizing or simplifying)

3. Special Cases

THEOREM 1.9 TWO SPECIAL TRIGONOMETRIC LIMITS

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ **2.** $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Ex 1

$$\lim_{x \to 2} \frac{\sqrt{x+2}}{x-4} = \frac{2}{-2}$$

$$= -1$$

Ex 2

$$\lim_{x \to 7} \sec\left(\frac{\pi x}{6}\right)$$

$$-\frac{2}{\sqrt{3}}$$

Ex 3

$$\lim_{x \to 5\pi/3} \cos x$$

$$\frac{1}{2}$$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

$\tan^2 x + 1 = \sec^2 x$

Other Trig identities

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$$\frac{\frac{1}{x+2}-\frac{1}{2}}{x}=\frac{\frac{2-(x+2)}{2(x+2)}}{\left(\frac{x}{1}\right)}$$

$$\frac{\cancel{2}-x-\cancel{2}}{2(x+2)}\cdot\frac{1}{x}$$

$$\frac{-1}{2(x+2)}$$

$$\frac{\left(\frac{1}{x+2}-\frac{1}{2}\right)}{\left(\frac{x}{1}\right)}\frac{\cdot 2(x+2)}{\cdot 2(x+2)}$$

$$\frac{2-(x+2)}{2x(x+2)}=\frac{-1}{2(x+2)}$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\frac{\cos 2x - 1}{-2} = \frac{\cancel{-2}\sin^2 x}{\cancel{-2}}$$

$$\frac{1 - \cos 2x}{2} = \sin^2 x$$

Ex 4

$$\lim_{x\to 2} \frac{3x^2-4x-4}{2x^2-8}$$

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(3x+2)}{2\cancel{(x-2)}(x+2)}$$

$$1$$

Ex 5

$$\lim_{x\to 2} \frac{x^3-8}{x^2-6x+8}$$

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x-4)}$$

$$\frac{12}{-2}$$

$$-6$$

Ex 6

$$\lim_{x\to\frac{\pi}{3}} \frac{2\cos^2 x + 3\cos x - 2}{2\cos x - 1}$$

$$\lim_{x\to\frac{\pi}{3}} \frac{\cancel{(2\cos x - 1)}(\cos x + 2)}{\cancel{2\cos x - 1}}$$

$$\frac{1}{2} + 2$$

$$\frac{5}{2}$$

Ex 7

$$\lim_{x\to 0} \frac{\cot x}{\csc x}$$

$$\lim_{x\to 0} \frac{\cos x}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{1}$$

$$1$$

Ex 8

$$\lim_{x\to 0}\frac{e^x-e^{2x}}{1-e^x}$$

$$\lim_{x\to 0}\frac{e^x\cancel{(1-e^x)}}{\cancel{1-e^x}}$$

$$1$$

Ex 9

$$\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16}\cdot\frac{(\sqrt{x}+4)}{(\sqrt{x}+4)}$$

$$\lim_{x\to 16}\frac{\cancel{x-16}}{\cancel{(x-16)}(\sqrt{x}+4)}$$

$$\frac{1}{8}$$

Ex 10

$$\lim_{x\to3}\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)}$$

$$\lim_{x\to3}\frac{\cancel{x+1-4}^{\,1}}{\cancel{(x-3)}(\sqrt{x+1}+2)}$$

$$\frac{1}{4}$$

11.) $\lim_{x \to 2} \frac{x-2}{\sqrt{x}-\sqrt{4-x}} \cdot \frac{(\sqrt{x}+\sqrt{4-x})}{(\sqrt{x}+\sqrt{4-x})}$

$\lim_{x \to 2} \frac{\cancel{(x-2)}(\sqrt{x}+\sqrt{4-x})}{\cancel{x-(4-x)}}$

$2\cancel{(x-2)}$

$\frac{\sqrt{2}+\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$:)

Ex 12

$$\lim_{x \to 4} \frac{\dfrac{x}{x+1} - \dfrac{4}{5}}{x-4}$$

$$\lim_{x \to 4} \frac{\dfrac{5x - 4(x+1)}{5(x+1)}}{\dfrac{x-4}{1}}$$

$$\lim_{x \to 4} \frac{\cancel{x-4}}{5(x+1)} \cdot \frac{1}{\cancel{x-4}} = \frac{1}{25} = .04$$

Ex 13

$$\lim_{x\to 0} \frac{3(1-\cos x)}{x}$$

$$3 \lim_{x\to 0} \frac{1-\cos x}{x}$$

$$3 \cdot 0$$

$$0$$

Ex 14

$$\lim_{t\to 0} \frac{\sin(3t)}{2t}$$

$$\frac{1}{2} \lim_{t\to 0} \frac{\sin 3t}{t} \cdot \frac{3}{3}$$

$$\frac{3}{2}\left[\lim_{t\to 0} \frac{\sin 3t}{3t}\right]$$

$$\frac{3}{2} \cdot 1$$

$$3/2$$

$$\lim_{x \to 0} \frac{\sin 4x}{11x}$$

$$\frac{1}{11} \lim_{x \to 0} \frac{\sin 4x}{x} \cdot \frac{4}{4}$$

$$\frac{4}{11} \cdot \left[\lim_{x \to 0} \frac{\sin 4x}{4x} \right]$$

$$\frac{4}{11}$$

Ex 15: Given f(x) = 5x - 2, $f(x+\Delta x) = 5(x+\Delta x) - 2$

find $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

$$\lim_{\Delta x \to 0} \frac{5(x+\Delta x) - 2 - (5x-2)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{5\Delta x}{\Delta x}$$

$$5$$

16.) $f(x) = -x^2 + 4x$ $\quad f(x+h) = \underline{-(x+h)^2 + 4(x+h)}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{-(x+h)^2 + 4(x+h) - (-x^2 + 4x)}{h}$$

$$\lim_{h \to 0} \frac{-(\cancel{x^2} + 2xh + h^2) + \cancel{4x} + 4h + \cancel{x^2} - \cancel{4x}}{h}$$

$$\lim_{h \to 0} -(2x+h) + 4 = -2x + 4$$

17.) $f(x) = \frac{1}{x}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{\frac{h}{1}}$$

$$\lim_{h \to 0} \frac{x - (x+h)}{x(x+h)} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$-\frac{1}{x^2}$$

Analyze a limit with a table

Analyze a limit with a graph

Analyze a limit analytically
- *direct substitution, factoring, rationalizing, simplifying, special cases*

Ex. 11

$$\lim_{x \to c} f(x) = \tfrac{3}{2}$$

$$\lim_{x \to c} g(x) = \tfrac{1}{2}$$

(a) $\lim_{x \to c} [4f(x)]$

(b) $\lim_{x \to c} [f(x) + g(x)]$

(c) $\lim_{x \to c} [f(x)g(x)]$

(d) $\lim_{x \to c} \dfrac{f(x)}{g(x)}$