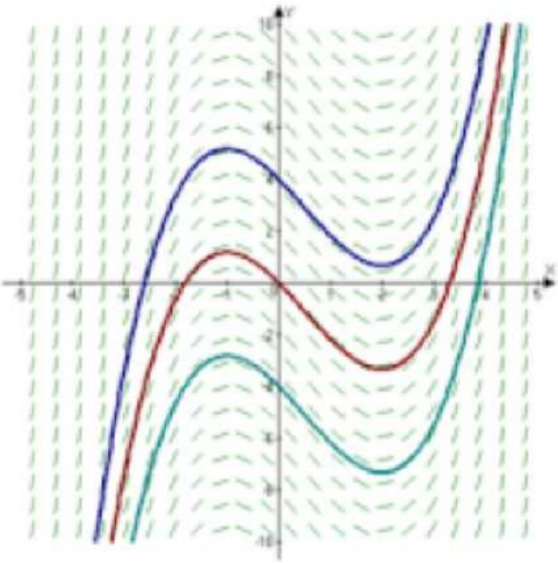
5.1/5.3 Slope Fields & Differential Equations

What is a Slope Field?



*See printout.

Sketching Slope Fields

1.

2.

Ex 1: Sketch each slope field.

a) $\dfrac{dy}{dx} = 2$

Ex 1: Sketch each slope field.

b) $\dfrac{dy}{dx} = x - 1$

Ex 1: Sketch each slope field.

c) $\frac{dy}{dx} = -3y$

Ex 1: Sketch each slope field.

d) $\dfrac{dy}{dx} = 2x + y$

Ex 1: Sketch each slope field.

e) $\frac{dy}{dx} = y + xy$

Ex 2: Consider the differential equaiton given by $\frac{dy}{dx} = \frac{x}{y}$.

a) Sketch a slope field.

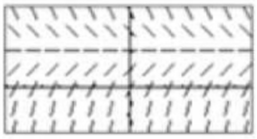
b) Sketch the solution curve that passes through the point (0, 1).

c) Sketch the solution curve that passes through the point (0, -1).

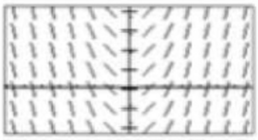
Matching Slope Fields with Differential Equations

Ex 3: Match each differential equation with the slope field.

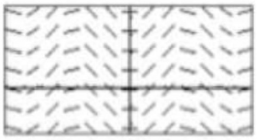
(A)



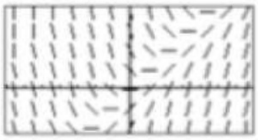
(B)



(C)



(D)



I. $\frac{dy}{dx} = \sin x$

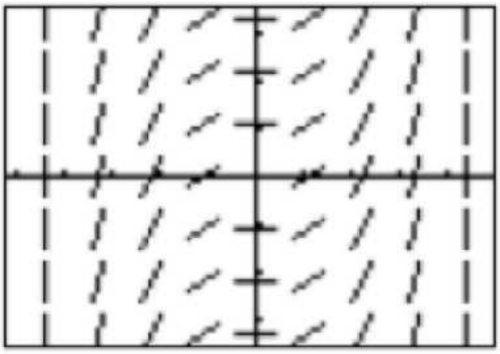
II. $\frac{dy}{dx} = x - y$

III. $\frac{dy}{dx} = 2 - y$

IV. $\frac{dy}{dx} = x$

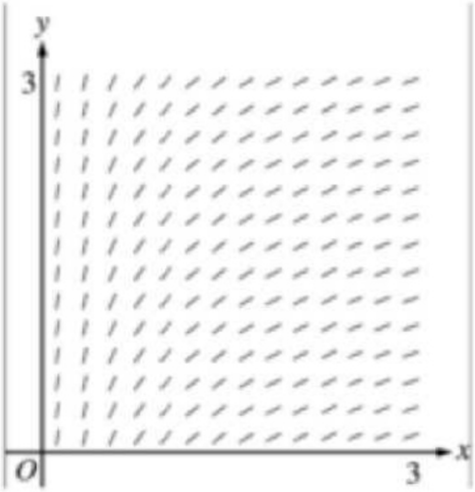
Matching Slope Fields with Equations

Ex 4: The slope field for a certain differential equation is shown below. Which of the following could be a particular solution to the differential equation?



(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = x^2$ (d) $y = \frac{1}{6}x^3$ (e) $y = \ln x$

Ex 5: The slope field for a certain differential equation is shown below. Which of the following could be a particular solution to the differential equation?



(a) $y = x^2$ (b) $y = e^x$ (c) $y = e^{-x}$ (d) $y = \cos x$ (e) $y = \ln x$

Ex 5: Verify the solution of the differential equation.

a)

Solution	Differential Equation
$y = e^{-2x}$	$3y' + 5y = -e^{-2x}$

$y' = -2e^{-2x}$

$3(-2e^{-2x}) + 5(e^{-2x}) = -e^{-2x}$

$-e^{-2x} = -e^{-2x}$

Ex 5: Verify the solution of the differential equation.

b)

Solution	Differential Equation
$y = 3\cos x + \sin x$	$y'' + y = 0$

$y' = -3\sin x + \cos x$

$y'' = -3\cos x - \sin x$

$-3\cos x - \sin x + 3\cos x + \sin x = 0$ ✓

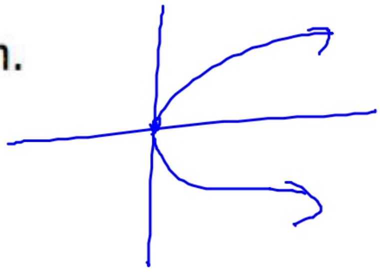
Two Types of Solutions to Differential Equations

1. *General solutions (+ C)*

2. *Particular solutions (solve for C)*

Ex 7: Find the general solution.

a) $y' = \dfrac{2x}{y}$



$$\frac{dy}{dx} = \frac{2x}{y}$$

$$\int y\,dy = \int 2x\,dx$$

$$\frac{y^2}{2} = x^2 + C$$

$$y^2 = 2x^2 + C$$

$$y = \pm\sqrt{2x^2 + C}$$

Ex 7: Find the general solution.

b) $y' = 3y$

$$\frac{dy}{dx} = 3y$$

$$\int \frac{dy}{y} = \int 3\,dx$$

$$\ln|y| = 3x + C$$

$$x^{2+3} = x^2 \cdot x^3$$

$$|y| = e^{3x+C}$$

$$|y| = e^{3x} \cdot e^{C}$$

$$|y| = Ce^{3x}$$

$$y = Ce^{3x}$$

$$|x| = 10$$

$$x = 10, \quad x = -10$$

Ex 8: Find the particular solution.

a) $y' = 7y$, (10, 1)

$$\frac{dy}{dx} = 7y$$

$$\int \frac{dy}{y} = \int 7\,dx$$

$$\ln|y| = 7x + C$$

$$0 = 70 + C$$

$$-70 = C$$

$$e^{\ln|y|} = e^{7x - 70}$$

$$|y| = e^{7x} \cdot e^{-70}$$

$$y = \frac{e^{7x}}{e^{70}}$$

$$y = e^{7x-70}$$

Ex 8: Find the particular solution.

b) $y' = \frac{x}{y}$, (0, -1)

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y^2 = x^2 + C$$

$$1 = 0 + C$$

$$y^2 = x^2 + 1$$

$$y = \pm\sqrt{x^2+1}$$

$$\boxed{y = -\sqrt{x^2+1}}$$

Ex 8: Find the particular solution.

c) $y' = \frac{y}{x^2}$, (1, 3)

$$\int \frac{dy}{y} = \int x^{-2}\,dx$$

$$\ln|y| = -\frac{1}{x} + C$$

$$\ln 3 = -1 + C$$

$$\ln 3 + 1 = C$$

$$e^{\ln|y|} = e^{-\frac{1}{x} + \ln 3 + 1}$$

$$|y| = e^{-1/x} \cdot e^{\ln 3} \cdot e^{1}$$

$$y = 3e^{-1/x + 1}$$

$$y = 3e^{1}e^{-1/x} = \frac{3e}{e^{1/x}}$$

Ex 8: Find the particular solution.

d) ~~$y\sqrt{1-x^2}\,y' - x\sqrt{1-y^2} = 0$, (0, 1)~~

$y' - xy^2 = 0$ $(3, 1)$

$y' = xy^2$

$\int \frac{dy}{y^2} = \int x\,dx$

$-\frac{1}{y} = \frac{x^2}{2} + C$

$-1 = \frac{9}{2} + C$

$-\frac{11}{2} = C$

$-\frac{1}{y} = \frac{x^2}{2} - \frac{11}{2}$

$-y = \frac{2}{x^2 - 11}$

$y = \frac{-2}{x^2 - 11}$

$y = \frac{2}{11 - x^2}$

Ex 9: The rate of change of y with respect to x is proportional to the difference between x and 4. Write a differential equation.

directly proportional

$y' = k(x-4)$

Ex 10: The rate of change of y with respect to x varies directly with the square of y. Write a differential equation.

$$y' = ky^2$$

The rate of change of y is proportional to y.

$$y' = ky$$

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln|y| = kt + C$$

$$e \qquad e$$

$$|y| = Ce^{kt}$$

$$y = Ce^{kt}$$

↑ general solution

The rate of change of y with respect to x is proportional to y. If y = 1 at x = 0 and y = 2 at x = 3, find x when y = 3.

$y = Ce^{kx}$

$c = 1$

$y = 1 \cdot e^{kx}$

$\ln 2 = \ln e^{3k}$

$\ln 2 = 3k$

$\frac{\ln 2}{3} = k$

$y = e^{\frac{\ln 2}{3} x}$

$y = (e^{\ln 2})^{\frac{x}{3}}$

$y = 2^{x/3}$

$3 = 2^{x/3}$

$3 = e^{\frac{\ln 2}{3} \cdot x}$

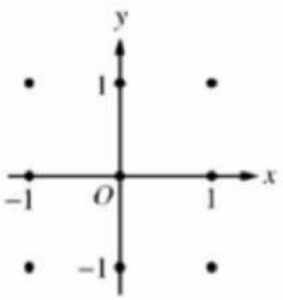
$\ln 3 = \frac{\ln 2}{3} \cdot x$

$\frac{3 \ln 3}{\ln 2} = x$

Ex 11:

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

Ex 12:

Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\frac{3x^2 + 1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

(c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$ with the initial condition $f(1) = 4$.

(d) Use your solution from part (c) to find $f(1.2)$.

Ex 13:

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

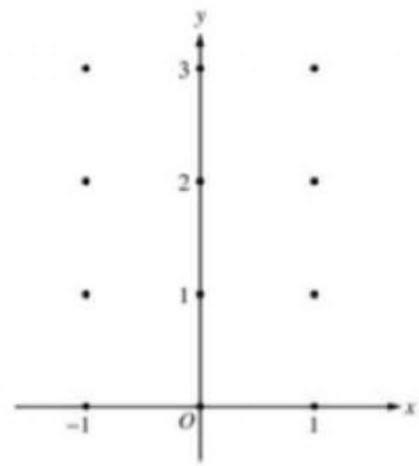
Ex 14:

Consider the differential equation $\frac{dy}{dx} = x^4(y - 2)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the test booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are negative.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 0$.



b.) $y < 2$ and $x \neq 0$

$$|y-2| = 2e^{x^5/5}$$

$$y-2 = 2e^{x^5/5} \quad \text{or} \quad y-2 = -2e^{x^5/5}$$

$$y = -2e^{x^5/5} + 2$$