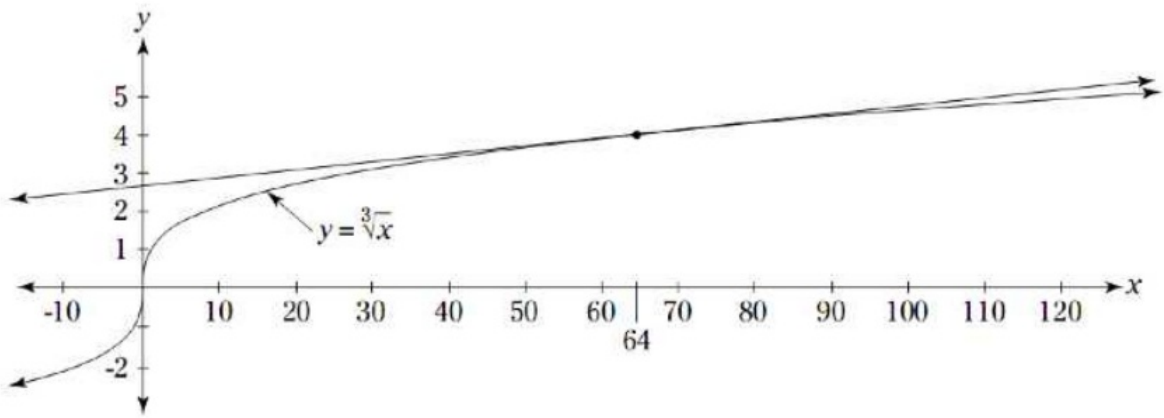
§3.7—Linearization & Differentials

Linear approximation is a very easy thing to do, and once you master it, you can impress all of your friends by calculating things like $\sqrt[3]{70}$ in your head . . . . about 4.125! Impressed? I'll teach you how.

Recall that if a function $f(x)$ is differentiable at $x=c$, we say it is locally linear at $x=c$. This means that as we zoom in closer and closer and closer and closer around $x=c$, the graph of $f(x)$, regardless of how curvy it is, will begin to look more and more and more and more like the tangent line at $x=c$.

This means that we can use the equation of the tangent line of $f(x)$ at $x=c$ to approximate $f(c)$ for values close to $x=c$. Let's take a look at $\sqrt[3]{70}$ and the figure below.



Example 1:
Approximate $\sqrt[3]{70}$ by using a tangent line approximation centered at $x=64$. Determine if this approximation is an over or under-approximation. Approximate $\sqrt[3]{70}$ using a secant line approximation using $x=64$ and $x=125$. Determine if this approximation is an over or under-approximation.

$f(x)=\sqrt[3]{x}$ $(64, 4)$

$f'(x)=\frac{1}{3}x^{-2/3}$

$f'(64)=\frac{1}{3}(64)^{-2/3}$

$=\frac{1}{3}\cdot\frac{1}{16}=\frac{1}{48}$

$f''(x)=-\frac{2}{9}x^{-5/3}$

$y-4=\frac{1}{48}(x-64)$

$y-4=\frac{1}{48}(70-64)$

$y=\frac{1}{8}+4=4\frac{1}{8}$

$\downarrow$

4.125

$\sqrt[3]{70}=4.1212853$

How to find linear approximations of $f(x)$ at $x = c$, the center to approximate $f(x)$ at $x = a$, a value near the center $x = c$.

1. Find the equation of the tangent line at the center $(c, f(c))$ in point-slope form.
2. Solve for y and rename it $L(x)$.
3. Plug in $x = a$ into $L(x)$ writing the notation VERY CAREFULLY as $f(a) \approx L(a) =$
4. If asked, determine if $L(a)$ is an over-approximation or an under approximation by examining the concavity of $f(x)$ at the center $x = c$.
   a. If $f''(c) < 0$, $f(x)$ is concave down at $x = c$ then $L(a)$ is an over-approximation
   b. If $f''(c) > 0$, $f(x)$ is concave up at $x = c$ and $L(a)$ is an under-approximation

Example 2: $\sqrt[4]{17}$

Estimate the fourth root of 17. Determine if the linearization is and over- or under-approximation.

$f(x) = \sqrt[4]{x}$

$f'(x) = \frac{1}{4}x^{-3/4}$

$f'(16) = \frac{1}{32}$

$(16, 2)$

$y - 2 = \frac{1}{32}(17 - 16)$

$= 2 + \frac{1}{32} = 2\frac{1}{32}$

$f''(16) < 0$

CCD $\therefore$

over-approx.

Example 3:

Approximate 3.01^5. Determine if the linearization is and over- or under-approximation.

$f(x) = x^5$

$f'(x) = 5x^4$

$f'(3) = 405$

$(3, 243)$

$y - 243 = 405(3.01 - 3)$

$y = 247.05$

$f''(x) = 20x^3$

$f''(3) > 0$

$\therefore$ CCU

under ✓

Example 4:

Approximate $\ln\left(e^{10} + 5\right)$. Determine if the linearization is and over- or under-approximation.

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f'(e^{10}) = \frac{1}{e^{10}}$

$(e^{10}, 10)$

$y - 10 = \frac{1}{e^{10}}(5) = \frac{5}{e^{10}} + 10$

$f''(e^{10}) < 0$

$\therefore$ CCD

over