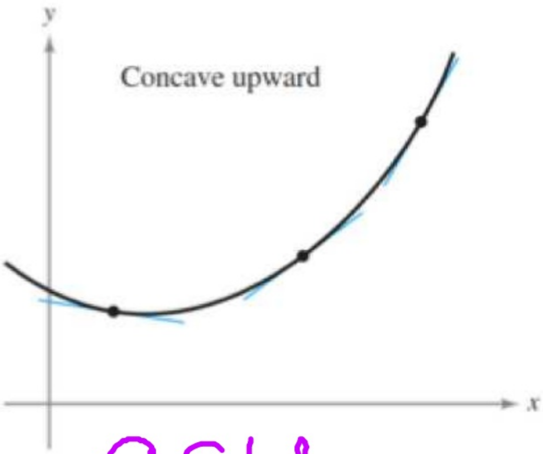
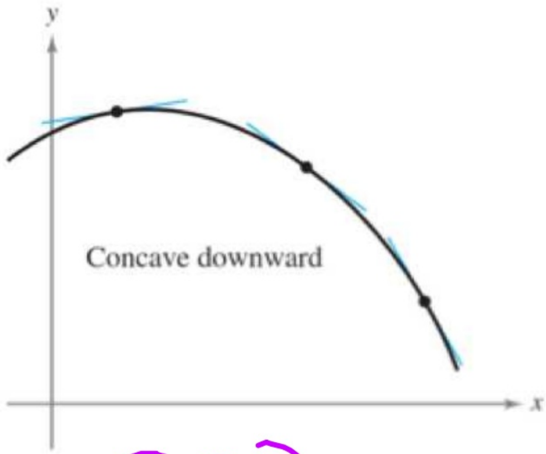
3.4 Concavity and the Second Derivative Test



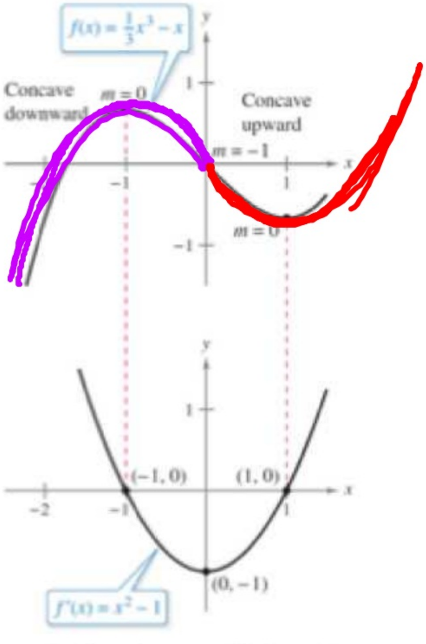


CCU
(cup)

CCD

Definition of Concavity

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I when f' is increasing on the interval and **concave downward** on I when f' is decreasing on the interval.



f' is decreasing. f' is increasing.

f	CCU	CCD
f'	incr.	decr.
f''	+	−

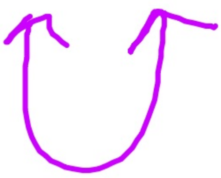
$y = x^2$

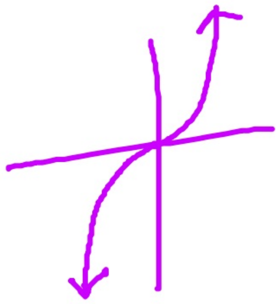
$y' = 2x$

$y'' = 2$

$y'' > 0$

CCU





$y = x^3$

$y' = 3x^2$

$y'' = 6x$

y'': $-$ | $+$

CCD 0 CCU

ex: Determine the open intervals on which the graph of

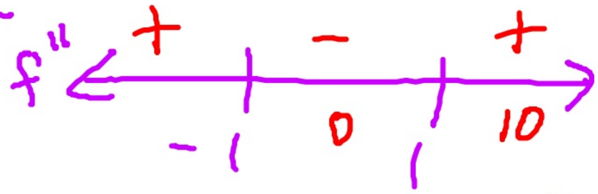
$$f(x) = e^{-\frac{x^2}{2}}$$

is concave upward and concave downward? Justify your answer.

$f'(x) = -xe^{-x^2/2}$

$f''(x) = e^{-x^2/2}(x^2-1)$

$x = \pm 1$



CCU $(-\infty,-1) \cup (1,\infty)$ because $f'' > 0$ on those intervals.

CCD $(-1,1)$ because $f'' < 0$ on this interval ex: Determine the open intervals on which the graph of

$$f(x) = \frac{x^2+1}{x^2-4}$$

is concave upward and concave downward? Justify your answer.

$$f'(x) = \frac{-10x}{(x^2-4)^2} \qquad f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3}$$

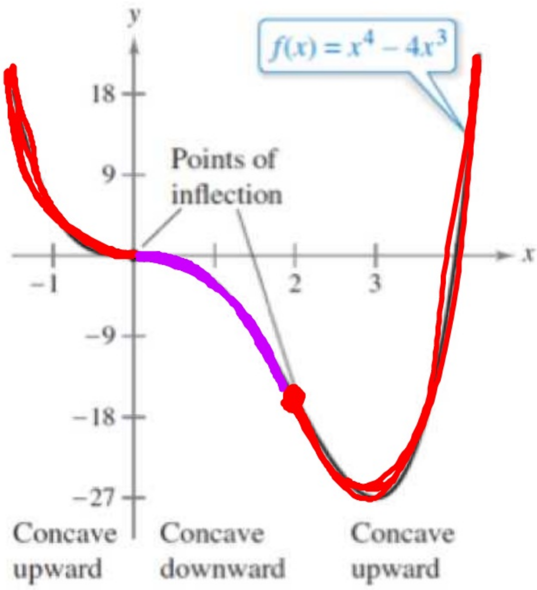
	$+$		$-$		$+$	f''
-10	-2	0	2	10		

CCU: $(-\infty, -2) \cup (2, \infty)$

CCD: $(-2, 2)$

Definition of Point of Inflection

Let f be a function that is continuous on an open interval, and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f when the concavity of f changes from upward to downward (or downward to upward) at the point.

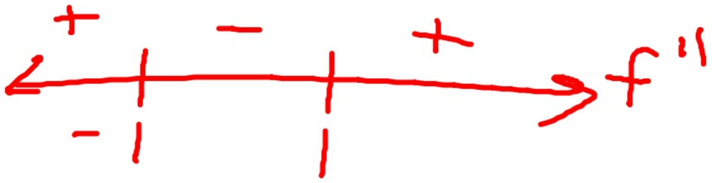


POI REQUIREMENTS

1. f(c) is defined
2. f''(c) is 0 or undefined
3. f'' changes signs at x=c.

ex: Find all points of inflection on the graph of f(x), if possible.

$$f(x)=e^{-\frac{x^2}{2}}$$



POI (-1, $e^{-1/2}$) and (1, $e^{-1/2}$) because f'' changes signs at these points.

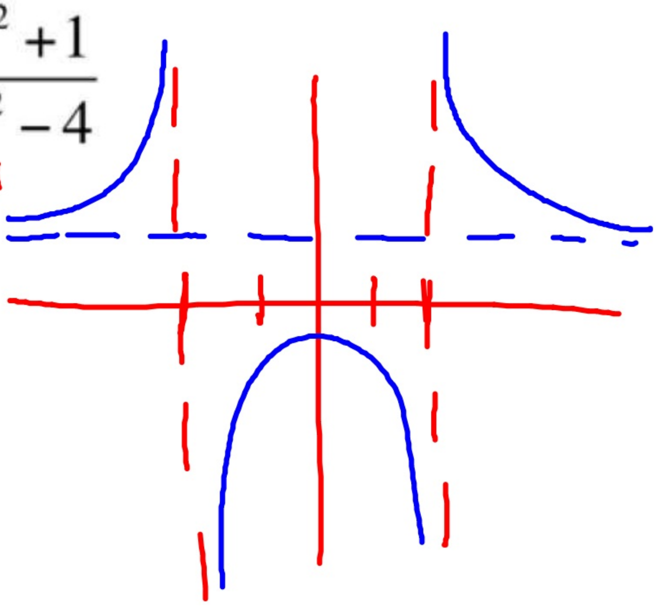
ex: Find all points of inflection on the graph of f(x), if possible.

$$f(x) = \frac{x^2+1}{x^2-4}$$

+ − +

-2 2 f''

NO POI

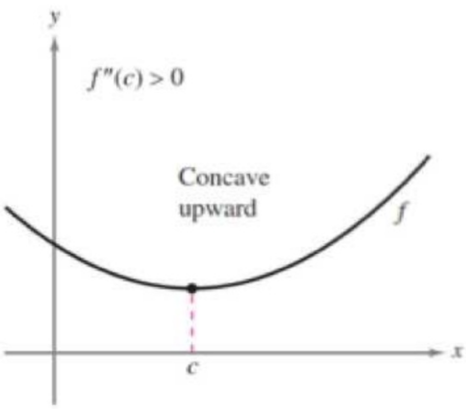


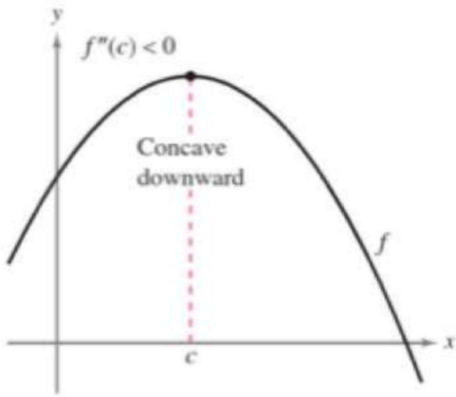
THEOREM 3.9 Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$. $\cup$ CCU
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$. $\cap$ CCD

If $f''(c) = 0$, then the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the ~~First Derivative Test.~~ not necessary





ex: Use the 2nd Derivative Test to find all relative extrema of

$$f(x) = -3x^5 + 5x^3$$

zeros of $f'(x)$

$f'(x) = -15x^4 + 15x^2$

$f'(x) = -15x^2(x^2 - 1)$

$x = 0, \pm 1$

$f''(x) = -60x^3 + 30x$

$f''(x) = -30x(2x^2 - 1)$

$f''(0) = 0$ no rel. ext.

$f''(1) < 0$ rel. max $(1, 2)$

$f''(-1) > 0$ rel. min $(-1, -2)$

ex: What can be concluded about f(x) at x=2 if

$$f(2) = 16$$

$$f'(2) = 0$$

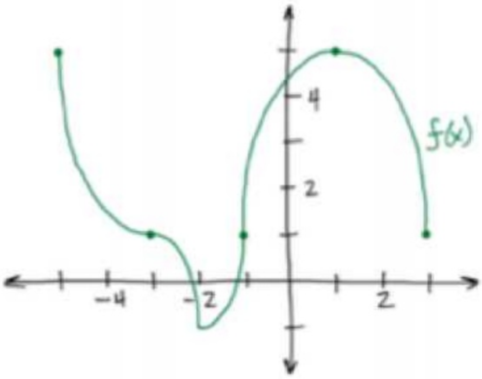
$$f''(2) = -300$$

(2,16): rel. max

3.4 WKST

2.

Let f be a continuous function on $[-5,3]$ with a vertical tangent line at $x=-1$, horizontal tangents at $x=-3$ and $x=1$ and a cusp at $x=-2$. The graph of f is given at right. Which of the following properties are satisfied?



I. $f''(x)<0$ on $(-2,1)$ ✗

II. f has exactly 2 local extrema ✓

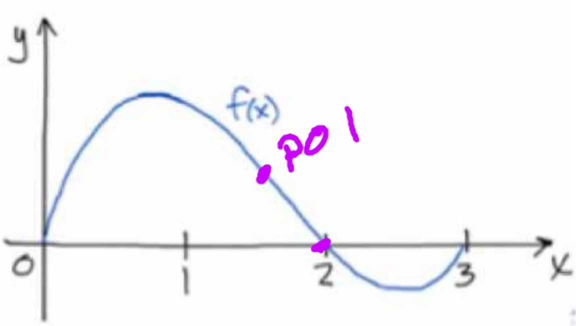
III. f has exactly 4 critical points ✓

(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

∩ ∪

3.4 WKST

4.



The graph of a differentiable function $f(x)$ is shown in the figure above and has an inflection point at $x = \frac{3}{2}$. Which of the following correctly orders $f(2)$, $f'(2)$, and $f''(2)$?

0 (−) (+)

(A) $f(2) < f'(2) < f''(2)$

(B) $f'(2) < f(2) < f''(2)$

(C) $f'(2) < f''(2) < f(2)$

(D) $f''(2) < f(2) < f'(2)$

(E) $f''(2) < f'(2) < f(2)$

3.4 WKST

6.

Let f be the function defined by $f(x)=2x^3-3x^2-12x+18$. On which of the following intervals is the graph of f both increasing and concave down?

(A) $(-\infty,-1)$ (B) $\left(-1,\frac{1}{2}\right)$ (C) $(-1,2)$ (D) $\left(\frac{1}{2},2\right)$ (E) $(2,\infty)$

3.4 WKST

7.

f' decr.

If $f'(x) > 0$ for all x and $f''(x) < 0$ for all x, which of the following could be a table of values for f?

(A)

x	$f(x)$
-1	4
0	3
1	1

(B)

x	$f(x)$
-1	4
0	4
1	4

(C)

x	$f(x)$
-1	4
0	5
1	6

(D)

x	$f(x)$
-1	4
0	5
1	7

(E)

x	$f(x)$
-1	4
0	6
1	7

3.4 WKST

8.

(**Calculator Permitted**) The derivative of the function f is given by $f'(x) = x^2 \sin\left(x^2\right)$. How many points of inflection does the graph of f have on the open interval $(-2, 2)$?

(A) One (B) Two (C) Three (D) Four (E) Five

3.4 WKST

9.

The second derivative of a function g is given by $g''(x) = 2^{-x^2} + \cos x + x$. For $-5 < x < 5$, on what open intervals is the graph of g concave up?

(A) $-5 < x < -1.016$ only

(B) $-1.016 < x < 5$ only

(C) $0.463 < x < 2.100$ only

(D) $-5 < x < 0.463$ and $2.100 < x < 5$