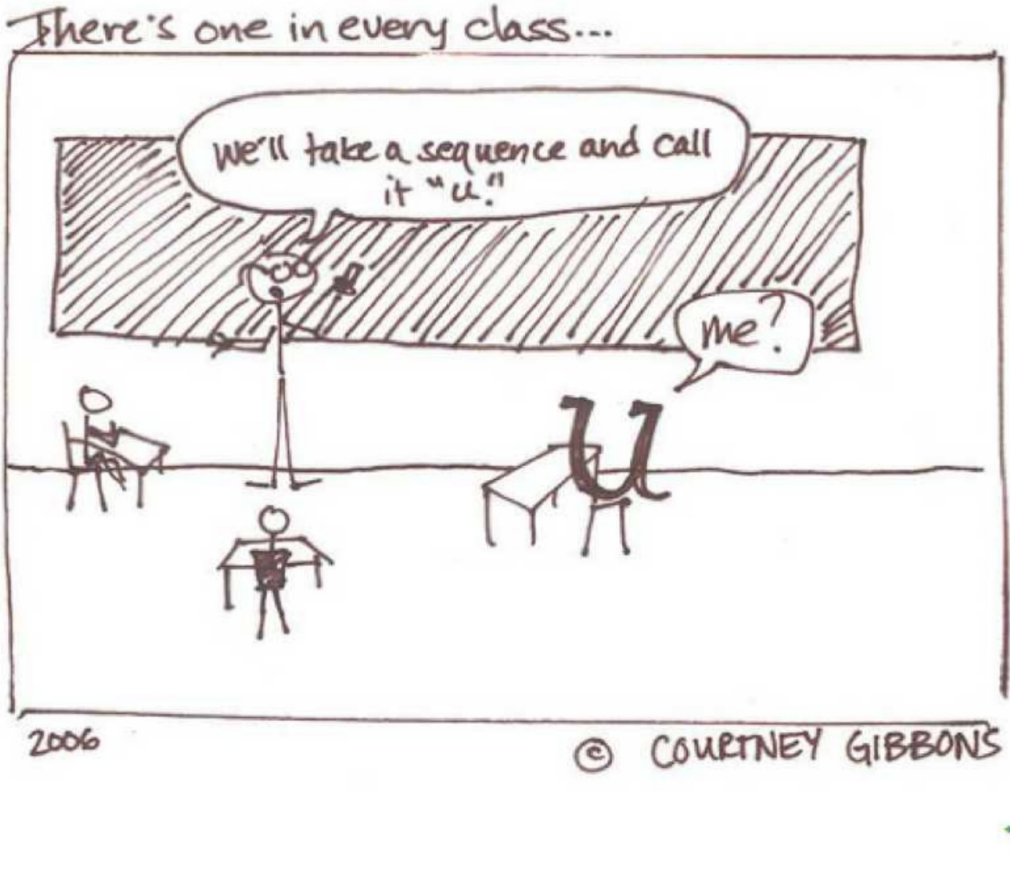
7.2 Arithmetic Sequences and Series



*See printout.

HW:

Definition of A Sequence

Sequences

A **sequence** is a function whose domain is a set of consecutive integers. If a domain is not specified, it is understood that the domain starts with 1. The values in the range are called the **terms** of the sequence.

$$\begin{array}{llllll l} \textbf{Domain:} & 1 & 2 & 3 & 4 \ldots & n & \text{The relative position of each term} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \textbf{Range:} & a_1 & a_2 & a_3 & a_4 \ldots & a_n & \text{Terms of the sequence} \end{array}$$

A *finite sequence* has a limited number of terms. An *infinite sequence* continues without stopping.

Finite sequence: 2, 4, 6, 8 **Infinite sequence:** 2, 4, 6, 8, . . .

A sequence can be specified by an equation, or *rule*. For example, both sequences above can be described by the rule $a_n = 2n$

Definition of A Series

Series

When the terms of a sequence are added together, the resulting expression is a **series**. A series can be finite or infinite.

Finite series: $2 + 4 + 6 + 8$ **Infinite series:** $2 + 4 + 6 + 8 + \cdots$

*In other words, a series is the sum of a sequence.

Notation

n "term #"

a_n "n^{th} term"

$a_n = 2n+5$

$a_3 = 2(3)+5$
$= 11$

Types of Sequences & Series

- Arithmetic
- Geometric
- etc.

Rule

A rule is a formula used to generate the terms of a sequence or series

Rules can be explicit or recursive. For example:

- Explicit: $a_n = 7n - 1$
- Recursive: $a_1 = 3, \; a_n = a_{n-1} - 2$

previous term

$a_1 = 3$

$a_n = a_{n-1} - 2$

$a_2 = a_1 - 2 = 1$

$a_3 = a_2 - 2 = -1$

ex: Find $\cancel{a_3}$. a_4

a) $a_n = 7n - 1$

$a_4 = 7(4) - 1$
$= 27$

b) $a_1 = 3$, $a_n = a_{n-1} - 2$

$a_2 = 3 - 2 = 1$
$a_3 = 1 - 2 = -1$
$a_4 = -1 - 2 = -3$

Arithmetic Sequences

In an **arithmetic sequence**, the difference of consecutive terms is constant. This constant difference is called the **common difference** and is denoted by *d*.

ex: Determine if the sequences is arithmetic. If so, identify the common difference.

a) 1, 2, 3, 4, 5 . . . Yes ; d = 1

b) 1, 1, 2, 3, 5 . . . No

c) 3, 0, -3, -6, -9 Yes ; d = -3

ex: Write the 1st ~~3~~ 4 terms of the sequence and sketch the graph.

$a_n = 1 - 2n$

$a_1 = -1$

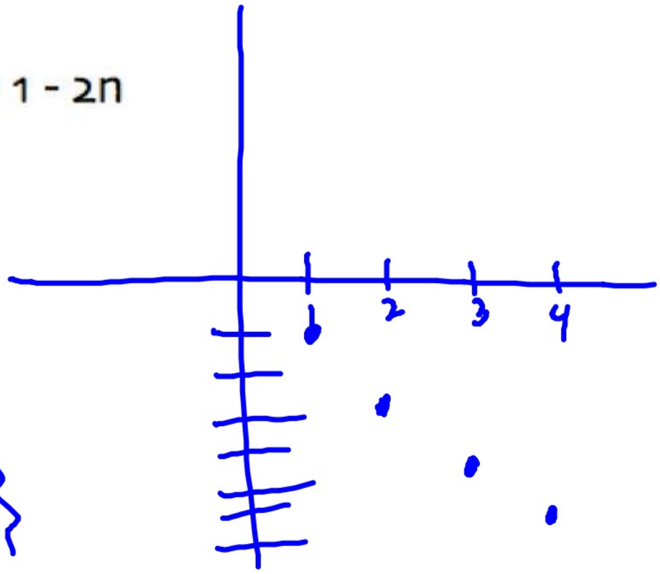
$a_2 = -3$

$a_3 = -5$

$a_4 = -7$

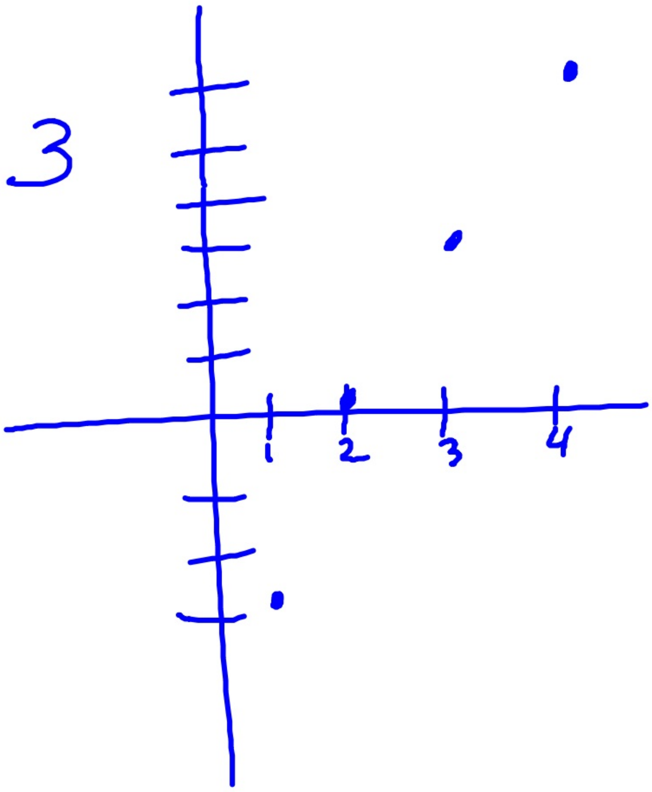
D: {1, 2, 3, 4}

R: {-1, -3, -5, -7}



$$a_1 = -3$$
$$a_n = a_{n-1} + 3$$

$$a_2 = 0$$
$$a_3 = 3$$
$$a_4 = 6$$



Writing Explicit Rules for Arithmetic Sequences/Series

*Since arithmetic sequences have a linear pattern, the explicit rule is linear!

Recall Point-Slope: $y - y_1 = m(x - x_1)$

$$\frac{y-y_1}{x-x_1} = m$$

Remember:

$n = x$

$a_n = y$

$d = m$

$$a_n = a_1 + (n-1)d$$

$\boxed{\text{Explicit Rule: } a_n - a_\# = d(n - n_\#)}$

ex: Write an explicit rule for the arithmetic sequence.

$(1, 20)$

a) 20, 30, 40, 50 . . .

a_1

$n = 1$

$d = 10$

$a_n - a_\# = d(n - n_\#)$

$a_n - 20 = 10(n - 1)$

$a_n = 10n - 10 + 20$

$a_n = 10n + 10$

b) 2, -2, -6, -10, -14 . . .

$a_1 = 2 \quad (1, 2)$

$a_n - 2 = -4(n - 1)$

$a_n = -4n + 6$

ex: Write an excplit rule for the arithmetic sequence.

c) $a_6 = 7$, $d = 9$

$(6, 7)$

$a_n - 7 = 9(n - 6)$

$a_n = 9n - 47$

ex: Write an excplit rule for the arithmetic sequence.

d) $a_{10} = -5$, $a_{20} = 75$

$(10, -5)$

$(20, 75)$

$a_n - a_{\#} = d(n - n_{\#})$

$75 - (-5) = d(20 - 10)$

$\frac{80}{10} = d$

$8 = d$

$a_n - (-5) = 8(n - 10)$

$a_n = 8n - 85$

Writing Recursive Rules for Arithmetic Sequences/Series

*Recursive rules give the beginning term or terms of a sequence and an equation that shows how a_n is related to one or more previous terms.

ex: Write a recursive rule for the arithmetic sequence.

a) 20, 30, 40, 50 . . .

$$\boxed{\begin{aligned} a_1 &= 20 \\ a_n &= a_{n-1} + 10 \end{aligned}}$$

ex: Write a recursive rule for the arithmetic sequence.

b) 2, -2, -6, -10, -14 . . .

$a_1 = 2$

$a_n = a_{n-1} - 4$

$1 + 2 + 3 + 4 + \ldots\ldots + 98 + 99 + 100$

$101(50)$

$$S_n = \frac{\overset{100}{n}(\overset{1}{a_1} + \overset{100}{a_n})}{2}$$

The Sum of a FINITE Arithmetic Sequence

The Sum of a Finite Arithmetic Series

The sum of the first n terms of an arithmetic series is:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

S_n sum of the 1^{st} n terms

n number of terms in the sum

a_1 1^{st} term in the sequence

a_n last term in the sequnce

*Infinite arithmetic sequences have "no sum." In other words the sum of an infinite arithmetic sequence is infinity or negative infinity and the sum diverges.

ex: Find the indicated sum, if possible.

a) $a_n = 12n + 15,\ S_{20} = ?$

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$n = 20$

$a_1 = 27$

$a_{20} = 255$

$$S_{20} = \frac{20(27+255)}{2}$$

$$= 2820$$

ex: Find the indicated sum, if possible.

b) $5 + 1 - 3 - 7 + \ldots \qquad S_{15} = ?$

$a_n - 5 = -4(n-1)$

$a_n = -4n + 9$

$n = 15$

$a_1 = 5$

$a_{15} = -51$

$S_{15} = \dfrac{15(5-51)}{2}$

$= -345$

ex: Find the indicated sum, if possible.

c) $2 + 6 + 10 + \ldots + \underline{58}$

$a_n - 2 = 4(n-1)$

$a_n = 4n - 2$

$58 = 4n - 2$

$15 = n$

$n = \underline{\quad}$

$a_1 = 2$

$a_{\boxed{15}} = 58$

$$S_{15} = \frac{15(2+58)}{2} = 450$$

ex: Find the indicated sum, if possible.

d) $2 + 6 + 10 + 14 + \ . \ . \ .$

infinite arithmetic sum

diverges

(no sum)

Summation Notation

Summation Notation (a.k.a. Sigma Notation) is used to express a series.

Upper Limit (where to stop)

Sigma $\longrightarrow \displaystyle\sum_{n=1}^{4} 3n+5$

Summand (Explicit Rule)

Lower Limit (where to begin)

ex: Find the sum, if possible.

a) $\sum_{n=1}^{15}(3n+5)$

$n = 15$

$a_1 = 8$

$a_{15} = 50$

$S_{15} = \frac{15(8+50)}{2}$

435

b) $\sum_{i=0}^{\infty} i-2$

no sum

$$\sum_{n=1}^{3} (n^2+1)$$

$$2 + 5 + 10$$

$$17$$

ex: Express the series using summation notation. Then find the sum or explain why there is no sum.

a) 3, 0, -3, -6, -9

rule: $a_n = (-3n + 6)$

$$\sum_{n=1}^{5} (-3n+6) = -15$$

ex: Express the series using summation notation. Then find the sum or explain why there is no sum.

b) $1 + 2 + 3 + 4 + 5 \ldots$

$a_n = n$

$$\sum_{n=1}^{\infty} n$$

no sum

ex: Express the series using summation notation. Then find the sum or explain why there is no sum.

c) 6 + 2 - 2 - 6 . . . - 78

$a_n = -4n + 10$

$-78 = -4n + 10$

$22 = n$

$$\sum_{n=1}^{22} (-4n+10) = -792$$

$a_1 = 6 \quad a_{22} = -78$

$n = 22$

$S_{22} = -792$