FRQ #1b Answers (#3 – #6)

3)

(a) $f\left(\frac{1}{2}\right) = 1$

$f'(x) = 24x^2$, so $f'\left(\frac{1}{2}\right) = 6$

An equation for the tangent line is $y = 1 + 6\left(x - \frac{1}{2}\right)$.

$2 : \begin{cases} 1 : f'\left(\frac{1}{2}\right) \\ 1 : \text{answer} \end{cases}$

(b) Area $= \int_0^{1/2} (g(x) - f(x))\,dx$

$= \int_0^{1/2} \left(\sin(\pi x) - 8x^3\right) dx$

$= \left[-\frac{1}{\pi}\cos(\pi x) - 2x^4\right]_{x=0}^{x=1/2}$

$= -\frac{1}{8} + \frac{1}{\pi}$

$4 : \begin{cases} 1 : \text{integrand} \\ 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $\pi\int_0^{1/2} \left((1 - f(x))^2 - (1 - g(x))^2\right) dx$

$= \pi\int_0^{1/2} \left(\left(1 - 8x^3\right)^2 - (1 - \sin(\pi x))^2\right) dx$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$

4)

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt = -6 - \frac{9\pi}{4}$ $g'(x) = 2 + f(x)$ $g'(-3) = 2 + f(-3) = 2$	$3: \begin{cases} 1: g(-3) \\ 1: g'(x) \\ 1: g'(-3) \end{cases}$
(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$. $g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$. Therefore g has an absolute maximum at $x = \frac{5}{2}$.	$3: \begin{cases} 1: \text{considers } g'(x) = 0 \\ 1: \text{identifies interior candidate} \\ 1: \text{answer with justification} \end{cases}$
(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of g has a point of inflection at $x = 0$.	1 : answer with reason
(d) The average rate of change of f on the interval $-4 \le x \le 3$ is $\frac{f(3) - f(-4)}{3 - (-4)} = -\frac{2}{7}$. To apply the Mean Value Theorem, f must be differentiable at each point in the interval $-4 < x < 3$. However, f is not differentiable at $x = -3$ and $x = 0$.	$2: \begin{cases} 1: \text{average rate of change} \\ 1: \text{explanation} \end{cases}$

5)

(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$.

The answer in part (a) is an underestimate.

$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \leq t \leq 20$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

6)

(a) $\lim_{x \to 0^-} (1 - 2\sin x) = 1$

$\lim_{x \to 0^+} e^{-4x} = 1$

$f(0) = 1$

So, $\lim_{x \to 0} f(x) = f(0)$.

Therefore f is continuous at $x = 0$.

2 : analysis

(b) $f'(x) = \begin{cases} -2\cos x & \text{for } x < 0 \\ -4e^{-4x} & \text{for } x > 0 \end{cases}$

$-2\cos x \neq -3$ for all values of $x < 0$.

$-4e^{-4x} = -3$ when $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right) > 0$.

Therefore $f'(x) = -3$ for $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right)$.

$3 : \begin{cases} 2 : f'(x) \\ 1 : \text{value of } x \end{cases}$

(c) $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$

$= \int_{-1}^{0} (1 - 2\sin x)\,dx + \int_{0}^{1} e^{-4x}\,dx$

$= \Big[x + 2\cos x\Big]_{x=-1}^{x=0} + \Big[-\frac{1}{4}e^{-4x}\Big]_{x=0}^{x=1}$

$= (3 - 2\cos(-1)) + \left(-\frac{1}{4}e^{-4} + \frac{1}{4}\right)$

Average value $= \frac{1}{2}\int_{-1}^{1} f(x)\,dx$

$= \frac{13}{8} - \cos(-1) - \frac{1}{8}e^{-4}$

$4 : \begin{cases} 1 : \int_{-1}^{0} (1 - 2\sin x)\,dx \text{ and } \int_{0}^{1} e^{-4x}\,dx \\ 2 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$