Euler's Method & Solving Differential Equations Notes

Ex 1: Find the general solution.

a) $y' = \dfrac{2x}{y}$

b) $y' = 3y$

c) $y' = x(1+y)$

d) $y' = 4 - x$

e) $y'\sqrt{9-x^2} = 5$

f) $y' = \dfrac{\sqrt{x}}{3y}$

Ex 2: Find particular solution that satisfies the given initial condition.

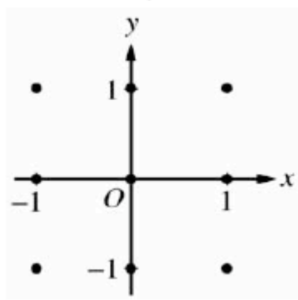
a) $y' = \dfrac{y}{x^2}$ $\quad (1,-3)$

b) $\left(x^2+1\right)y' = \dfrac{x}{y}$ $\quad \left(\sqrt{5},6\right)$

Ex 3:

Consider the differential equation $\dfrac{dy}{dx} = (y-1)^2\cos(\pi x)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

Ex 4:

Given $y' = \dfrac{-2x}{y}$, use Euler's Method, starting at the point $(0,1)$ with a step size of 0.2 to approximate $f(0.4)$.

Ex 5:

Use Euler's Method to approximate the solution of the differential equation $y' = x - y$ that passes through the point $(0,1)$. Use a step size of 0.1 and three steps.

Ex 6:

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim\limits_{x\to\infty} f(x) = 0$.

(a) Evaluate $\displaystyle\int_1^{\infty} -3xf(x)\,dx$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

(c) Write an expression for $y = f(x)$ by solving the differential equation $\dfrac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

Ex 7: (CALCULATOR)

Let f be the function whose graph goes through the point $(3, 6)$ and whose derivative is given by $f'(x) = \dfrac{1 + e^x}{x^2}$.

(a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.

(b) Use Euler's method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.

(c) Use $\displaystyle\int_3^{3.1} f'(x)\,dx$ to evaluate $f(3.1)$.

Ex 8:

Consider the differential equation $\dfrac{dy}{dx} = 3x + 2y + 1$.

a.

Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

b.

Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = -2$. Use Euler's method, starting at $x = 0$ with a step size of $\frac{1}{2}$, to approximate $f(1)$. Show the work that leads to your answer.

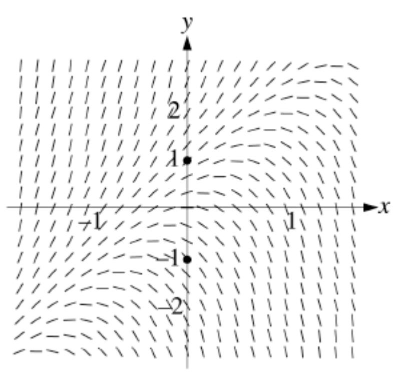
c.

Let $y = g(x)$ be another solution to the differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 1, gives the approximation $g(1) \approx 0$. Find the value of k.

Ex 9:

Consider the differential equation $\dfrac{dy}{dx} = 2y - 4x$.

(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0, 1)$ and sketch the solution curve that passes through the point $(0, -1)$.
(Note: Use the slope field provided in the pink test booklet.)



(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0, 0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

ANSWERS

1. a. $y = \pm\sqrt{2x^2 + c}$ b. $y = Ae^{3x}$ c. $y = Ae^{\frac{1}{2}x^2} - 1$ d. $y = 4x - \frac{1}{2}x^2 + c$

   e. $y = \arcsin\left(\frac{x}{3}\right) + c$ f. $y = \pm\sqrt{\frac{4}{9}x^{\frac{3}{2}} + c}$

2. a. $y = -3e^{1-\frac{1}{x}}$ b. $y = \sqrt{\ln(x^2+1) + \frac{36}{\ln 6}}$

3. a. graph
   b. y=1
   c. $y = 1 - \dfrac{1}{\frac{1}{\pi}\sin(\pi x) + 1}$

4. $y(1) \approx \dfrac{23}{25}$

5. $y(1.3) \approx .758$

6. a. $\int_1^\infty -3xf(x)dx = -4$
   b. $f(2) \approx \dfrac{5}{2}$
   c. $y = 4e^{-\frac{3}{2}x^2 + \frac{3}{2}}$

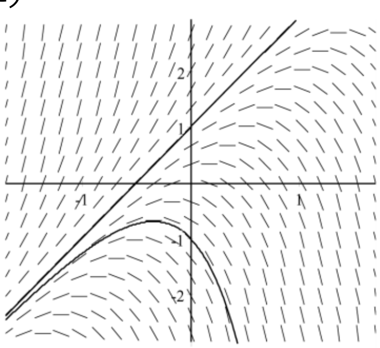
7. a. $T(x) = 6 + \dfrac{1+e^3}{9}(x-3),$ $f(3.1) \approx T(3.1) = 6.234$
   b. $f(3.1) \approx L(3.1) = 6.236$

   $f''(x) = \dfrac{e^x x^2 - 2x(1+e^x)}{x^4} > 0 \; on \; [3, 3.1], \therefore L(3.1) < f(3.1)$

   c. $f(3.1) = 6 + \int_3^{3.1} f'(x)dx \approx 6.238$

8. a. $\dfrac{d^2y}{dx^2} = 3 + 2(3x + 2y + 1)$
   b. $f(1) \approx -\dfrac{23}{4}$
   c. $k = -\dfrac{1}{3}$

9.
a)



b) $f(0.2) \approx 1.4$

c)
Substitute $y = 2x + b$ in the DE:
$2 = 2(2x + b) - 4x = 2b$, so $b = 1$

OR

Guess $b = 1$, $y = 2x + 1$

Verify: $2y - 4x = (4x + 2) - 4x = 2 = \frac{dy}{dx}$.

d) g has a local maximum at (0,0) by the 2nd derivative test because $g'(0) = 0$ (g has a critical number at x=0) and $g''(0) < 0$ (g is concave down at x=0).