1. Let $f(x) = x^3 + 9x$ on $[1,3]$.

(a) What is the average rate of change in f on $[1,3]$?

(b) On what interval(s) is $f'(x) > 0$?

(c) Find a value c in the interval $[1,3]$ such that the average rate of change in f over the interval $[1,3]$ equals $f'(c)$.

1. Let $f(x) = x^3 + 9x$ on $[1,3]$.

(a) What is the average rate of change in f on $[1,3]$? average rate $= \dfrac{f(3) - f(1)}{3-1}$ $= \dfrac{54-10}{2}$ $= 22$	+1 Use difference quotient +1 Answer is 22
(b) On what interval(s) is $f'(x) > 0$? $f'(x) = 3x^2 + 9$ $= 3(x^2 + 1)$ Since $x^2 + 1 > 0$ for all x, $f'(x) > 0$ for all x. Thus $f'(x) > 0$ on the interval $[1,3]$.	+1 $f'(x) = 3x^2 + 9$ +1 Justify why $f'(x) > 0$ for all x. +1 The interval $[1,3]$
(c) Find a value c in the interval $[1,3]$ such that the average rate of change in f over the interval $[1,3]$ equals $f'(c)$. $f'(c) = 3c^2 + 9$ $22 = 3c^2 + 9$ $13 = 3c^2$ $c^2 = \dfrac{13}{3}$ $c = \sqrt{\dfrac{13}{3}}$ ≈ 2.082	+1 $f'(c) = 3c^2 + 9$ +1 Set $f'(c)$ equal to average rate of change from (a) or equal to 22 +2 $c = \sqrt{\dfrac{13}{3}} \approx 2.082$ (-1 if $-\sqrt{\dfrac{13}{3}}$)

2. Let $f(x) = \sin x - \cos x$ on $[0, \pi]$.

(a) What is the average rate of change in f on $[0, \pi]$?

(b) On what interval(s) is $f'(x) > 0$?

(c) Find a value c in the interval $[0, \pi]$ such that the average rate of change in f over the interval $[0, \pi]$ equals $f'(c)$.

2. Let $f(x) = \sin x - \cos x$ on $[0, \pi]$.

(a) What is the average rate of change in f on $[0,\pi]$? average rate $= \dfrac{f(\pi)-f(0)}{\pi-0}$ $= \dfrac{\sin(\pi)-\cos(\pi)-(\sin(0)-\cos(0))}{\pi}$ $= \dfrac{0-(-1)-(0-1)}{\pi}$ $= \dfrac{2}{\pi}$	+1 Use difference quotient +1 Answer is $\dfrac{2}{\pi}$
(b) On what interval(s) is $f'(x) > 0$? $g'(x) = \cos x + \sin x$ on $[0,\pi]$. Since the coordinate for angle x on the unit circle is $(\cos x, \sin x)$, $g'(x) = \cos x + \sin x$ is the sum of the coordinate values for angle x. In the first quadrant, both coordinate values are positive so $g'(x) > 0$ for $0 < x < \frac{\pi}{2}$. At $x = 0$ and $x = \frac{\pi}{2}$, the coordinates are $(1,0)$ and $(0,1)$, respectively, so $g'(x) > 0$ at those values also. In the second quadrant, $\cos x < 0$. Additionally, $\sin x > \lvert\cos x\rvert$ for $\frac{\pi}{2} < x < \frac{3\pi}{4}$. Therefore, $\cos x + \sin x > 0$ on $\frac{\pi}{2} < x < \frac{3\pi}{4}$. Thus $g'(x) > 0$ for $0 < x < \frac{3\pi}{4}$.	+1 $g'(x) = \cos x + \sin x$ +1 $g'(x) > 0$ for $0 \le x \le \frac{\pi}{2}$ with valid argument +1 $g'(x) > 0$ for $\frac{\pi}{2} < x < \frac{3\pi}{4}$ with valid argument
(c) Find a value c in the interval $[0,\pi]$ such that the average rate of change in f over the interval $[0,\pi]$ equals $f'(c)$. $\cos c + \sin c = \dfrac{2}{\pi}$ Solving with technology, $x \approx 1.889$.	+1 $f'(c) = \cos c + \sin c$ +1 Set $f'(c)$ equal to average rate of change from (a) or equal to 22 +2 $x \approx 1.889$

3. Let $g(x) = x - \sin(\pi x)$ on $[0, 2]$.

(a) Find the critical value(s) of g.

(b) What are the coordinates of the relative extrema ? Justify your conclusion.

(c) At what value of x does g have an inflection point? Justify your conclusion.

3. Let $g(x) = x - \sin(\pi x)$ on $[0,2]$.

(a) Find the critical value(s) of g. $g'(x) = 1 - \pi\cos(\pi x)$ $0 = 1 - \pi\cos(\pi x)$ $\pi\cos(\pi x) = 1$ $\cos(\pi x) = \frac{1}{\pi}$ $\pi x = \cos^{-1}\left(\frac{1}{\pi}\right)$ $x = \frac{\cos^{-1}\left(\frac{1}{\pi}\right)}{\pi}$ $\approx 0.397,\ 1.603$	+1 $g'(x) = 1 - \pi\cos(\pi x)$ +1 Set $g'(x) = 0$ +1 $x = 0.397,\ 1.603$
(b) What are the coordinates of the relative extrema? Justify your conclusion. From (a), we know a critical value occurs at $x = 0.397$. Since $g'(0) < 0$, $g'(0.397) = 0$, and $g'(1) > 0$, a relative minimum occurs at $x = 0.397$. To find the coordinates of the point, we evaluate $g(0.397)$. $g(0.397) = -0.551$. Thus the coordinates of the relative minimum are $(0.397, -0.551)$. From (a), we know a critical value occurs at $x = 1.603$. Since $g'(1) > 0$, $g'(1.603) = 0$, and $g'(2) < 0$, a relative maximum occurs at $x = 1.603$. To find the coordinates of the point, we evaluate $g(1.603)$. $g(1.603) = 2.551$. Thus the coordinates of the relative maximum are $(1.603, 2.551)$.	+1 Show g' changes from negative to positive at $x = 0.397$. +1 Show g' changes from positive to negative at $x = 1.603$. +1 A relative minimum is $(0.397, -0.551)$ +1 A relative maxima is $(1.603, 2.551)$.
(c) At what value of x does g have an inflection point? Justify your conclusion. $g''(x) = \pi^2\sin(\pi x)$ $0 = \pi^2\sin(\pi x)$ $0 = \sin(\pi x)$ $\sin^{-1}(0) = \pi x$ $0, \pi = \pi x$ $x = 0$ and 1 On the interval $[0,2]$, g'' changes sign at $x = 1$ so an inflection point occurs at $x = 1$.	+1 $g''(x) = \pi^2\sin(\pi x)$ +1 $x = 1$ because g'' changes sign

4. Let $f(x)=\dfrac{2x}{x^2+1}$ on. $[-5,5]$.

(a) Find the critical value(s) of f.

(b) What are the coordinates of the relative extrema? Justify your conclusion.

(c) What is $\lim_{x\to\infty} f(x)$? Justify your solution.

4. Let $f(x)=\dfrac{2x}{x^2+1}$ on. $[-5,5]$.

(a) Find the critical value(s) of f. $f'(x)=\dfrac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$ $=\dfrac{-2x^2+2}{(x^2+1)^2}$ $0=\dfrac{-2(x-1)(x+1)}{(x^2+1)^2}$ $x=\pm 1$	+1 $f'(x)=\dfrac{-2x^2+2}{(x^2+1)^2}$ +1 Set $f'(x)=0$ +1 $x=\pm 1$
(b) What are the coordinates of the relative extrema? Justify your conclusion. We evaluate the derivative on both sides of the critical values and draw the following conclusions. $f'(x)<0$ for $x<-1$ $f'(-1)=0$ $f'(x)>0$ for $0<x<1$ $f'(1)=0$ $f'(x)<0$ for $x>1$ By the First Derivative Test, a relative minimum occurs at $x=-1$ and a relative maximum occurs at $x=1$. The corresponding coordinates are $(-1,-1)$ and $(1,1)$, respectively.	+1 Show f' changes from negative to positive at $x=-1$. +1 Show f' changes from positive to negative at $x=1$. +1 A relative minimum is $(-1,-1)$ +1 A relative maxima is $(1,1)$.
(c) What is $\lim_{x\to\infty} f(x)$? Justify your solution. As f approaches infinity, $f(x)=\dfrac{2x}{x^2+1}$ may be closely approximated by $y=\dfrac{2x}{x^2}=\dfrac{2}{x}$. We know that $\lim_{x\to\infty}\dfrac{2}{x}=0$. Therefore, $\lim_{x\to\infty} f(x)=0$.	+1 $\lim_{x\to\infty} f(x)=0$ +1 Justification

5. Let $f(x) = \dfrac{2x^2}{x^2 + 4}$ on. $[-5, 5]$.

(a) On what interval(s) is f increasing? Show the work that leads to your conclusion.

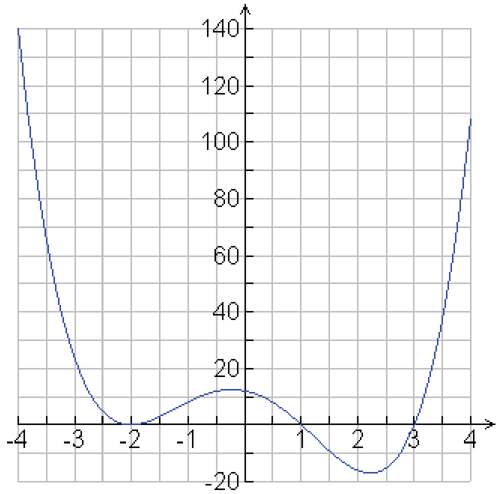
(b) On what interval(s) is f concave up? Justify your conclusion.

(c) At what values of x does f have an inflection point? Justify your conclusion.

5. Let $f(x)=\dfrac{2x^2}{x^2+4}$ on. $[-5,5]$.

(a) On what interval(s) is f increasing? Show the work that leads to your conclusion. $f'(x)=\dfrac{4x\left(x^2+4\right)-\left(2x^2\right)(2x)}{\left(x^2+4\right)^2}$ $=\dfrac{16x}{\left(x^2+4\right)^2}$ For positive values of x, $f'(x)>0$. Therefore, f is increasing on the interval $(0,5]$.	+1 $f'(x)=\dfrac{16x}{\left(x^2+4\right)^2}$ +1 $f'(x)>0$ for $x>0$ +1 increasing on $(0,5]$
(b) On what interval(s) is f concave up? Justify your conclusion. $f''(x)=\dfrac{16\left(x^2+4\right)^2-(16x)\left(2\left(x^2+4\right)(2x)\right)}{\left(x^2+4\right)^2}$ $=\dfrac{16\left(x^2+4\right)^2-64x^2\left(x^2+4\right)}{\left(x^2+4\right)^2}$ $=\dfrac{16\left(x^2+4\right)-64x^2}{x^2+4}$ $=\dfrac{-48x^2+64}{x^2+4}$ $=\dfrac{-16\left(3x^2-4\right)}{x^2+4}$ We observe that for $-\sqrt{\dfrac{4}{3}}<x<\sqrt{\dfrac{4}{3}}$, $f''(x)>0$. Therefore f is concave up on the open interval $\left(-\sqrt{\dfrac{4}{3}},\sqrt{\dfrac{4}{3}}\right)$.	+1 $f''(x)=\dfrac{-16\left(3x^2-4\right)}{x^2+4}$ +1 For $-\sqrt{\dfrac{4}{3}}<x<\sqrt{\dfrac{4}{3}}$, $f''(x)>0$. +1 Concave up on $\left(-\sqrt{\dfrac{4}{3}},\sqrt{\dfrac{4}{3}}\right)$
(c) At what values of x does f have an inflection point? Justify your conclusion. In part (b), we showed that for $-\sqrt{\dfrac{4}{3}}<x<\sqrt{\dfrac{4}{3}}$, $f''(x)>0$ and f is concave up. Similarly, $f''(x)<0$ and f is concave down for $x<-\sqrt{\dfrac{4}{3}}$ and $x>\sqrt{\dfrac{4}{3}}$. Since the concavity changes at $x\pm\sqrt{\dfrac{4}{3}}$, f has points of inflection there.	+1 $x=-\sqrt{3}$ +1 $x=\sqrt{3}$ +1 Justification

6. The graph of $f(t)$ is shown in the figure below.



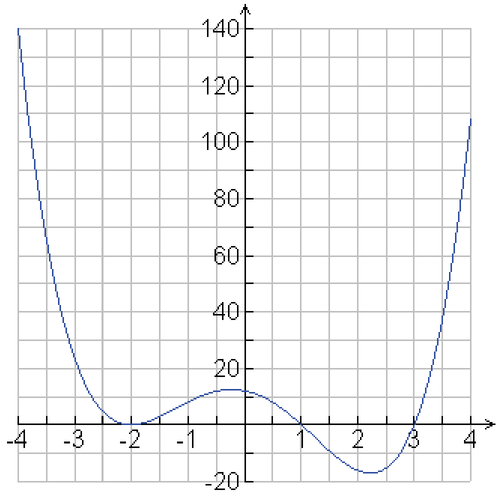
//Comp: Equation is $f(t)=(t-1)(t-3)(t+2)^2$

(a) Estimate from the graph the interval(s) on which f' is positive. Explain how you know.

(b) Estimate from the graph the interval(s) on which f'' is positive. Explain how you know.

(c) Estimate from the graph the values of t where $f'(t)=0$. Explain how you know.

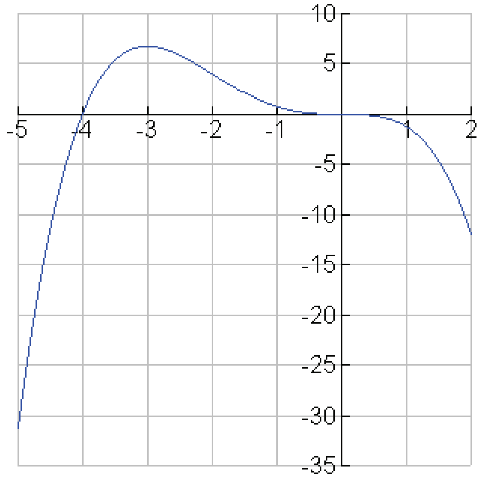
6. The graph of $f(t)$ is shown in the figure below.



//Comp: Equation is $s(t)=(t-1)(t-3)(t+2)^2$

(a) Estimate from the graph the interval(s) on which f' is positive. Explain how you know. Where the graph is increasing, f' is positive. The graph appears to be increasing on the intervals $-2<t<-0.5$ and $2.5<t<4$. Thus f' is positive on those intervals.	+1 $-2<t<-0.5$ +1 $2.5<t<4$ +1 Where the graph is increasing, f' is positive.
(b) Estimate from the graph the interval(s) on which f'' is positive. Explain how you know. Where the graph is concave up, f'' is positive. The graph appears to concave up on the intervals $-4<t<-1$ and $1<t<4$. Thus f'' is positive on those intervals.	+1 $-4<t<-1$ +1 $1<t<4$ +1 Where the graph is concave up, f'' is positive.
(c) Estimate from the graph the values of t where $f'(t)=0$. Explain how you know. Since the graph of f is continuous and differentiable, we need only determine where the graph has horizontal tangent lines. This will occur at a relative extrema. The relative extrema appear to occur at $t=-2,-0.5,$ and 2.5.	+1 $t=-2,-0.5,$ and 2.5 +2 Explanation

7. The graph of $f(x)$ is shown in the figure below.



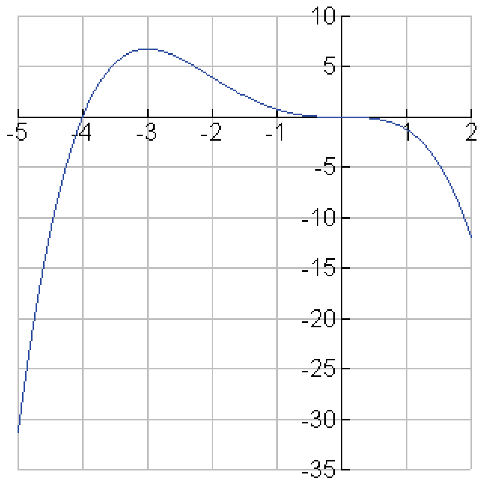
//Comp: Equation is $f(x) = -0.25x^4 - x^3$

(a) Estimate from the graph the interval(s) on which f' is positive. Explain how you know.

(b) Estimate from the graph the interval(s) on which f'' is negative. Explain how you know.

(c) Estimate from the graph the values of t where $f'(x) = 0$. Explain how you know.

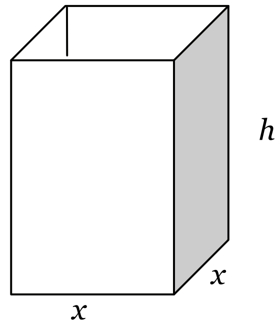
7. The graph of $f(x)$ is shown in the figure below.



//Comp: Equation is $f(x) = -0.25x^4 - x^3$

(a) Estimate from the graph the interval(s) on which f' is positive. Explain how you know. Where the graph is increasing, f' is positive. The graph appears to be increasing on the interval $-5 < x < -3$. Thus f' is positive on that interval.	+1 $-5 < x < -3$ +1 Where the graph is increasing, f' is positive.
(b) Estimate from the graph the interval(s) on which f'' is negative. Explain how you know. Where the graph is concave down, f'' is positive. The graph appears to concave down on the intervals $-5 < x < -2$ and $0 < x < 2$. Thus f'' is negative on those intervals.	+1 $-5 < x < -2$ +1 $0 < x < 2$ +1 Where the graph is concave up, f'' is positive.
(c) Estimate from the graph the values of t where $f'(x) = 0$. Explain how you know. Since the graph of f is continuous and differentiable, we need only determine where the graph has horizontal tangent lines. The graph appears to have horizontal tangents at $x = -3$ and $x = 0$.	+1 $x = -3$ +1 $x = 0$ +1 Horizontal tangents idea

8. An open rectangular box with square base has a volume of 256 cubic inches.

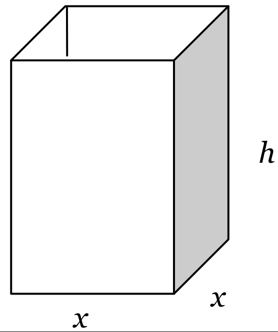


(a) Determine the equation for the volume and surface area of the box.

(b) What box dimensions minimize surface area?

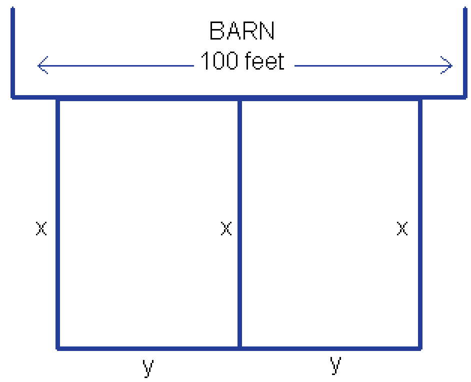
(c) With the added restriction that the box height cannot exceed 3 inches, what is the minimum surface area?

8. An open rectangular box with square base has a volume of 256 cubic inches.



<table>
<tr><td>(a) Determine the equation for the volume and surface area of the box.
$V = x^2h$ and $A = x^2 + 4xh$</td><td>+1 $V = x^2h$
+1 $A = x^2 + 4xh$</td></tr>
<tr><td>(b) What box dimensions minimize surface area?
$256 = x^2h$
$h = \frac{256}{x^2}$
$A = 4xh + x^2$
$= 4x\left(\frac{256}{x^2}\right) + x^2$
$= 1024x^{-1} + x^2$
$A' = -1024x^{-2} + 2x$
$0 = -\frac{1024}{x^2} + 2x$
$\frac{1024}{x^2} = 2x$
$1024 = 2x^3$
$x^3 = 512 \Rightarrow x = 8$
$h = \frac{256}{8^2}$
$= 4$
$A'(1) < 0 \Rightarrow A$ decreasing
$A'(8) = 0$
$A'(10) > 0 \Rightarrow A$ increasing
The minimum dimensions are 8 inches by 8 inches by 4 inches.</td><td>+1 $h = \frac{256}{x^2}$
+1 $A' = -1024x^{-2} + 2x$
+1 $x = 8$ inches
+1 $h = 4$ inches
+1 verify minimum</td></tr>
<tr><td>(c) With the added restriction that the box height cannot exceed 3 inches, what is the minimum surface area?
We know that A is decreasing on the interval $0 < x < 8$ and increasing for $x > 8$. Our objective is to pick x as close to 8 as possible. Since $256 = x^2h$, $x = \sqrt{\frac{256}{h}}$. As h increases, x decreases. Since the maximum value of h is 3, the minimum value of x is $x = \sqrt{\frac{256}{3}} \approx 9.238$. The minimum surface area is $A = \left(\sqrt{\frac{256}{3}}\right)^2 + 4\left(\sqrt{\frac{256}{3}}\right)(3) \approx 196.185$ sq in.</td><td>+1 $x = \sqrt{\frac{256}{3}} \approx 9.238$
+1 $A \approx 196.185$ sq in</td></tr>
</table>

9. Two pens are to be built alongside a barn as shown in the figure. The barn will make up one side of each pen.



(a) If 200 feet are available, what pen size maximizes area?

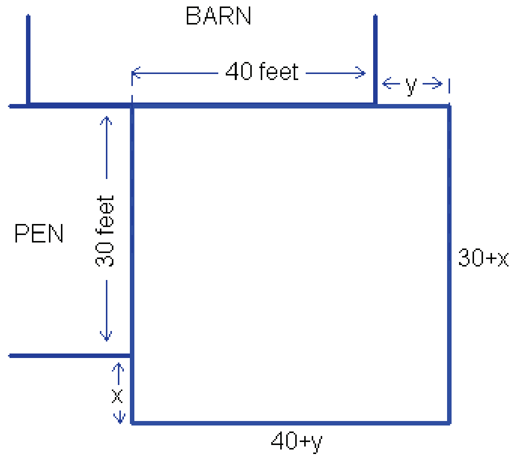
(b) If 500 feet are available, what pen size maximizes area?

(c) If s feet are available, what pen size maximizes area?

9. Two pens are to be built alongside a barn as shown in the figure. The barn will make up one side of each pen.

(a) If 200 feet are available, what pen size maximizes area? $3x+2y=200$ $2y=-3x+200$ $y=-1.5x+100$ $y=-1.5\left(33\tfrac{1}{3}\right)+100$ $=50$ feet $A=xy$ $=x\left(-1.5x+100\right)$ $=-1.5x^2+100x$ $A'=-3x+100$ $0=-3x+100$ $x=33\tfrac{1}{3}$ feet	+1 $A=-1.5x^2+100x$ +1 $x=33\tfrac{1}{3}$ feet +1 $y=50$ feet
(b) If 500 feet are available, what pen size maximizes area? $3x+2y=500$ $2y=-3x+500$ $y=-1.5x+250$ $y=-1.5\left(83\tfrac{1}{3}\right)+250$ $=125$ feet $A=xy$ $=x\left(-1.5x+250\right)$ $=-1.5x^2+250x$ $A'=-3x+250$ $0=-3x+250$ $x=83\tfrac{1}{3}$ feet But the length of the barn is only 100 feet so $2y\le 100 \Rightarrow y\le 50$. $50\ge -1.5x+250$ $-200\ge -1.5x$ $133\tfrac{1}{3}\le x$ For $x>83\tfrac{1}{3}$, A is decreasing. So we want the smallest value of x that meets the constraints. Thus $x=133\tfrac{1}{3}$ and $y=50$.	+1 Identify domain restriction $133\tfrac{1}{3}\le x$ +1 $x=133\tfrac{1}{3}$ +1 $y=50$
(c) If s feet are available, what pen size maximizes area? $3x+2y=s$ $2y=-3x+s$ $y=-1.5x+0.5s$ $y=-1.5\left(\tfrac{1}{6}s\right)+0.5s$ $=0.25s$ feet $A=xy$ $=x\left(-1.5x+0.5s\right)$ $=-1.5x^2+0.5sx$ $A'=-3x+0.5s$ $0=-3x+0.5s$ $x=\tfrac{1}{6}s$ feet Since $2y\le 100$, $y\le 50$. $50\ge -1.5x+0.5s$ $100\ge -3x+s$ $\dfrac{s-100}{3}\le x$ For $x>\tfrac{1}{6}s$, A is decreasing. If $\tfrac{1}{6}s\ge \dfrac{s-100}{3}$, $x=\tfrac{1}{6}s$. If $\tfrac{1}{6}s< \dfrac{s-100}{3}$, $x=\dfrac{s-100}{3}$	+1 $x=\tfrac{1}{6}s$ with correct condition +1 $x=\dfrac{s-100}{3}$ with correct condition +1 $y=-1.5x+0.5s$

10. A new pen is to be built between a barn and an existing pen as shown in the figure. The sides of the new pen bordered by the barn and the existing pen will not require any new fence.

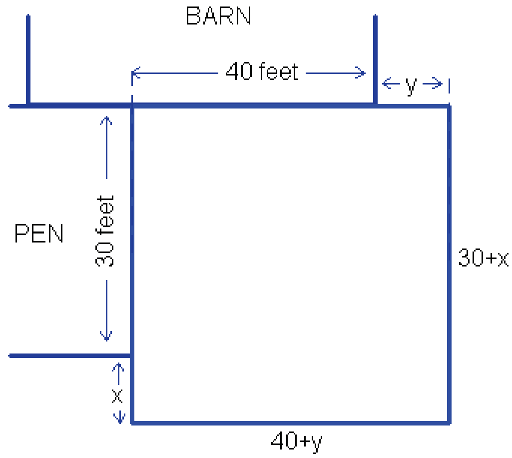


(a) Write an equation that represents the amount of fence needed to enclose the new pen.

(b) Write an equation that represents the area of the new pen.

(c) If the new pen needs to contain 1444 square feet of area, what pen size minimizes the amount of fence needed?

10. A new pen is to be built between a barn and an existing pen as shown in the figure. The sides of the new pen bordered by the barn and the existing pen will not require any new fence.



(a) Write an equation that represents the amount of fence needed to enclose the new pen. $P = x + 40 + y + 30 + x + y$ $= 2x + 2y + 70$	+1 $P = 2x + 2y + 70$
(b) Write an equation that represents the area of the new pen. $A = (40 + y)(30 + x)$	+1 $A = (40 + y)(30 + x)$
(c) If the new pen needs contain 1444 square feet of area, what pen size minimizes the amount of fence needed? $1444 = (40 + y)(30 + x)$ $x = \frac{1444}{40 + y} - 30$ $P = 2x + 2y + 70$ $= 2\left(\frac{1444}{40 + y} - 30\right) + 2y + 70$ $= \frac{2888}{40 + y} - 60 + 2y + 70$ $= \frac{2888}{40 + y} + 2y + 10$ $P' = -2888(40 + y)^{-2} + 2$ $0 = -2888(40 + y)^{-2} + 2$ $2 = 2888(40 + y)^{-2}$ $2(40 + y)^2 = 2888$ $(40 + y)^2 = 1444$ $40 + y = 38$ $y = -2$ $x = \frac{1444}{40 + (-2)} - 30$ $= 38 - 30$ $= 8$ $30 + 8 = 38$ $40 + (-2) = 38$ The pen should be 38 feet by 38 feet.	+1 $x = \frac{1444}{40 + y} - 30$ +1 $P = \frac{2888}{40 + y} + 2y + 10$ +1 $P' = -2888(40 + y)^{-2} + 2$ +1 Set $P' = 0$ +1 $y = -2$ +1 $x = 8$ +1 38 feet by 38 feet