

Ch 5/6 Free Response Questions

1. Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $\frac{dy}{dt} = ky$, where y is the amount of oil left in the well at any time t . Initially there were 1,000,000 gallons of oil in the well, and 6 years later there were 500,000 gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining.
- (a) Write an equation for y , the amount of oil remaining in the well at any time t .
- (b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?
- (c) In order not to lose money, at what time t should oil no longer be pumped from the well?

- 2.
- (a) Find the general solution of the differential equation $xy' + y = 0$.
- (b) Find the general solution of the differential equation $xy' + y = 2x^2y$.
- (c) Find the particular solution of the differential equation in part (b) that satisfies the condition that $y = e^2$ when $x = 1$.

3. Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\frac{3x^2 + 1}{2y}$.
- (a) Find the slope of the graph of f at the point where $x = 1$.
- (b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.
- (c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$ with the initial condition $f(1) = 4$.
- (d) Use your solution from part (c) to find $f(1.2)$.

4. Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.
- (a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.
- (b) Find the domain and range of the function f found in part (a).

5.

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

- (a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.
- (b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

6.

Consider the differential equation $\frac{dy}{dx} = \frac{3 - x}{y}$.

- (a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f . Find the x -coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.
- (b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

7.

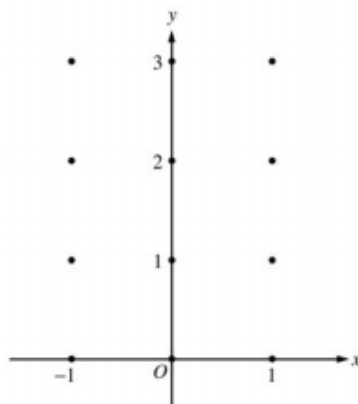
Let f be the function satisfying $f'(x) = x\sqrt{f(x)}$ for all real numbers x , where $f(3) = 25$.

- (a) Find $f''(3)$.
- (b) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = x\sqrt{y}$ with the initial condition $f(3) = 25$.

8.

Consider the differential equation $\frac{dy}{dx} = x^4(y - 2)$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the test booklet.)
- (b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy -plane. Describe all points in the xy -plane for which the slopes are negative.
- (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 0$.



9.

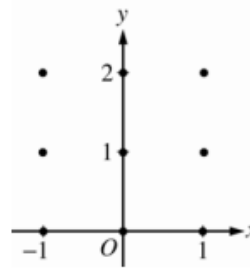
Consider the curve given by $y^2 = 2 + xy$.

- Show that $\frac{dy}{dx} = \frac{y}{2y-x}$.
- Find all points (x, y) on the curve where the line tangent to the curve has slope $\frac{1}{2}$.
- Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.
- Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time $t = 5$.

10.

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

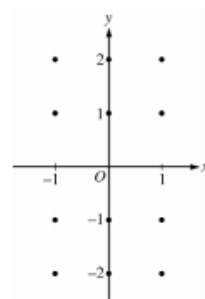
- On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)
- Find $\frac{d^2y}{dx^2}$ in terms of x and y . Describe the region in the xy -plane in which all solution curves to the differential equation are concave up.
- Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.
- Find the values of the constants m and b , for which $y = mx + b$ is a solution to the differential equation.



11.

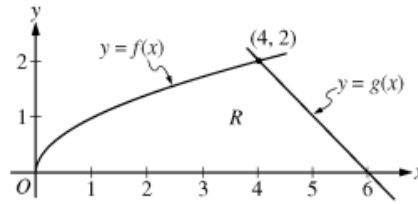
Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

- On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.
(Note: Use the axes provided in the exam booklet.)
- While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy -plane for which $y \neq 0$. Describe all points in the xy -plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.
- Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.



12.

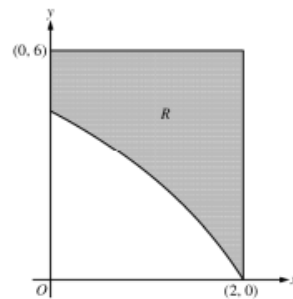
The functions f and g are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let R be the region bounded by the x -axis and the graphs of f and g , as shown in the figure above.



- Find the area of R .
- The region R is the base of a solid. For each y , where $0 \leq y \leq 2$, the cross section of the solid taken perpendicular to the y -axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.
- There is a point P on the graph of f at which the line tangent to the graph of f is perpendicular to the graph of g . Find the coordinates of point P .

13. CALCULATOR

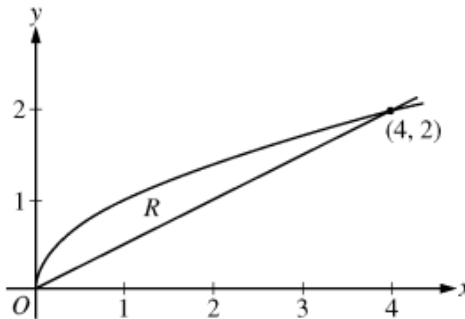
In the figure above, R is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.



- Find the area of R .
- Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.
- The region R is the base of a solid. For this solid, each cross section perpendicular to the x -axis is a square. Find the volume of the solid.

14.

Let R be the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$, as shown in the figure above.



- Find the area of R .
- The region R is the base of a solid. For this solid, the cross sections perpendicular to the x -axis are squares. Find the volume of this solid.
- Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 2$.

15.

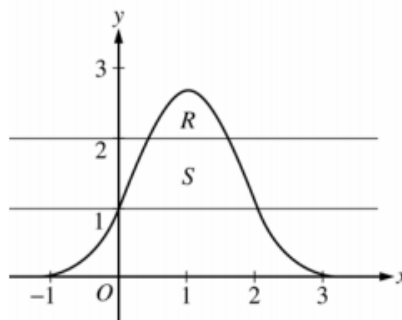
Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$.

- Find the area of R .
- Find the volume of the solid generated when R is rotated about the vertical line $x = -1$.
- The region R is the base of a solid. For this solid, the cross sections perpendicular to the y -axis are squares. Find the volume of this solid.

16. CALCULATOR

Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

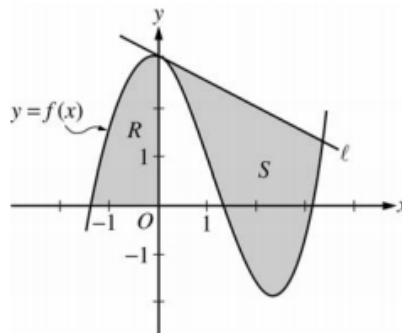
- Find the area of R .
- Find the area of S .
- Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 1$.



17. CALCULATOR

Let f be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let R be the shaded region in the second quadrant bounded by the graph of f , and let S be the shaded region bounded by the graph of f and line ℓ , the line tangent to the graph of f at $x = 0$, as shown above.

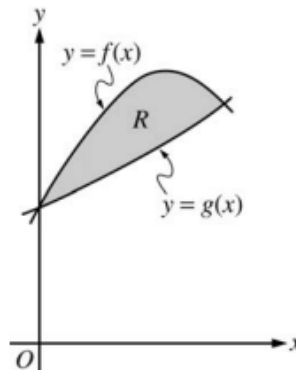
- Find the area of R .
- Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.
- Write, but do not evaluate, an integral expression that can be used to find the area of S .



18. CALCULATOR

Let f and g be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let R be the shaded region in the first quadrant enclosed by the graphs of f and g as shown in the figure above.

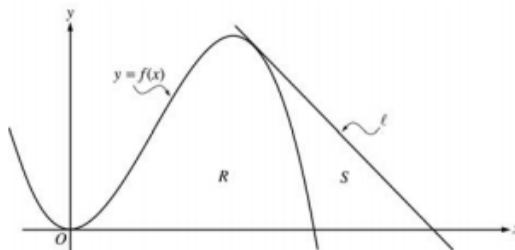
- Find the area of R .
- Find the volume of the solid generated when R is revolved about the x -axis.
- The region R is the base of a solid. For this solid, the cross sections perpendicular to the x -axis are semicircles with diameters extending from $y = f(x)$ to $y = g(x)$. Find the volume of this solid.



19. CALCULATOR

Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f . Let R be the region bounded by the graph of f and the x -axis, and let S be the region bounded by the graph of f , the line ℓ , and the x -axis, as shown above.

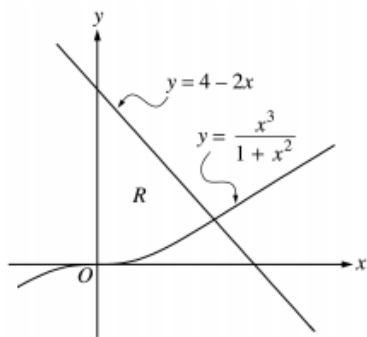
- Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.
- Find the area of S .
- Find the volume of the solid generated when R is revolved about the x -axis.



20. CALCULATOR

Let R be the region bounded by the y -axis and the graphs of $y = \frac{x^3}{1+x^2}$ and $y = 4 - 2x$, as shown in the figure above.

- Find the area of R .
- Find the volume of the solid generated when R is revolved about the x -axis.
- The region R is the base of a solid. For this solid, each cross section perpendicular to the x -axis is a square. Find the volume of this solid.



21. NO CALCULATOR (OMIT part (b))

Let R be the region enclosed by the graph of $y = \frac{x^2}{x^2 + 1}$, the line $x = 1$, and the x -axis.

- Find the area of R .
- Find the volume of the solid generated when R is rotated about the y -axis.

22. NO CALCULATOR (OMIT part (c))

Let R be the region enclosed by the graphs of $y = e^x$, $y = (x - 1)^2$, and the line $x = 1$.

- Find the area of R .
- Find the volume of the solid generated when R is revolved about the x -axis.
- Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y -axis.

ANSWERS

1.

$$(a) \begin{cases} \frac{dy}{dt} = ky \\ y = Ce^{kt} \end{cases} \quad \text{or} \quad \begin{cases} \frac{dy}{y} = k dt \\ \ln|y| = kt + C_1 \\ y = e^{kt+C_1} \end{cases}$$

$$t = 0 \Rightarrow C = 10^6, C_1 = \ln 10^6$$

$$\therefore y = 10^6 e^{kt}$$

$$t = 6 \Rightarrow \frac{1}{2} = e^{6k}$$

$$\therefore k = -\frac{\ln 2}{6}$$

$$y = 10^6 e^{6 \cdot \left(-\frac{\ln 2}{6}\right) t} = 10^6 \cdot 2^{-t}$$

$$(b) \frac{dy}{dt} = ky = -\frac{\ln 2}{6} \cdot 6 \cdot 10^5 \\ = -10^5 \ln 2$$

Decreasing at $10^5 \ln 2$ gal/year

$$(c) 5 \cdot 10^4 = 10^6 e^{kt}$$

$$\therefore kt = -\ln 20$$

$$\therefore t = \frac{-\ln 20}{-\frac{\ln 2}{6}}$$

$$= 6 \frac{\ln 20}{\ln 2} = 6 \log_2 20$$

$$6 \frac{\ln 20}{\ln 2} \text{ years after starting}$$

2.

$$a) y = A/x$$

$$b) y = Ae^{x^2 - \ln x}$$

$$c) y = \frac{e^{x^2+1}}{x}$$

3.

$$(a) \frac{dy}{dx} = \frac{3x^2 + 1}{2y}$$

$$\left. \frac{dy}{dx} \right|_{\substack{x=1 \\ y=4}} = \frac{3+1}{2 \cdot 4} = \frac{4}{8} = \frac{1}{2}$$

$$(b) y - 4 = \frac{1}{2}(x - 1)$$

$$f(1.2) - 4 \approx \frac{1}{2}(1.2 - 1)$$

$$f(1.2) \approx 0.1 + 4 = 4.1$$

$$(c) 2y dy = (3x^2 + 1) dx$$

$$\int 2y dy = \int (3x^2 + 1) dx$$

$$y^2 = x^3 + x + C$$

$$4^2 = 1 + 1 + C$$

$$14 = C$$

$$y^2 = x^3 + x + 14$$

$$y = \sqrt{x^3 + x + 14} \text{ is branch with point } (1, 4)$$

$$f(x) = \sqrt{x^3 + x + 14}$$

$$(d) f(1.2) = \sqrt{1.2^3 + 1.2 + 14} \approx 4.114$$

4.

$$(a) e^{2y} dy = 3x^2 dx$$

$$\frac{1}{2} e^{2y} = x^3 + C_1$$

$$e^{2y} = 2x^3 + C$$

$$y = \frac{1}{2} \ln(2x^3 + C)$$

$$\frac{1}{2} = \frac{1}{2} \ln(0 + C); C = e$$

$$y = \frac{1}{2} \ln(2x^3 + e)$$

$$(b) \text{Domain: } 2x^3 + e > 0$$

$$x^3 > -\frac{1}{2}e$$

$$x > \left(-\frac{1}{2}e\right)^{1/3} = -\left(\frac{1}{2}e\right)^{1/3}$$

$$\text{Range: } -\infty < y < \infty$$

5.

$$(a) \frac{d^2y}{dx^2} = 2y \frac{dy}{dx} (6-2x) - 2y^2$$

$$= 2y^3(6-2x)^2 - 2y^2$$

$$\frac{d^2y}{dx^2} \Big|_{\left(3, \frac{1}{4}\right)} = 0 - 2\left(\frac{1}{4}\right)^2 = -\frac{1}{8}$$

$$(b) \frac{1}{y^2} dy = (6-2x) dx$$

$$-\frac{1}{y} = 6x - x^2 + C$$

$$-4 = 18 - 9 + C = 9 + C$$

$$C = -13$$

$$y = \frac{1}{x^2 - 6x + 13}$$

6.

$$(a) \frac{dy}{dx} = 0 \text{ when } x = 3$$

$$\frac{d^2y}{dx^2} \Big|_{(3,-2)} = \frac{-y - y'(3-x)}{y^2} \Big|_{(3,-2)} = \frac{1}{2},$$

so f has a local minimum at this point.

or

Because f is continuous for $1 < x < 5$, there is an interval containing $x = 3$ on which

$y < 0$. On this interval, $\frac{dy}{dx}$ is negative to

the left of $x = 3$ and $\frac{dy}{dx}$ is positive to the

right of $x = 3$. Therefore f has a local

minimum at $x = 3$.

$$(b) y dy = (3-x) dx$$

$$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$$

$$8 = 18 - 18 + C; C = 8$$

$$y^2 = 6x - x^2 + 16$$

$$y = -\sqrt{6x - x^2 + 16}$$

7.

$$(a) f''(x) = \sqrt{f(x)} + x \cdot \frac{f'(x)}{2\sqrt{f(x)}} = \sqrt{f(x)} + \frac{x^2}{2}$$

$$f''(3) = \sqrt{25} + \frac{9}{2} = \frac{19}{2}$$

$$(b) \frac{1}{\sqrt{y}} dy = x dx$$

$$2\sqrt{y} = \frac{1}{2}x^2 + C$$

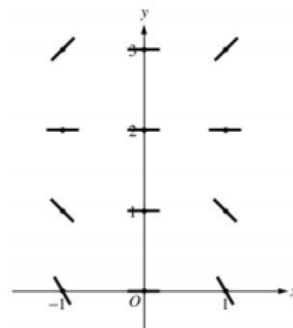
$$2\sqrt{25} = \frac{1}{2}(3)^2 + C; C = \frac{11}{2}$$

$$\sqrt{y} = \frac{1}{4}x^2 + \frac{11}{4}$$

$$y = \left(\frac{1}{4}x^2 + \frac{11}{4}\right)^2 = \frac{1}{16}(x^2 + 11)^2$$

8.

(a)



(b) Slopes are negative at points (x, y) where $x \neq 0$ and $y < 2$.

$$(c) \frac{1}{y-2} dy = x^4 dx$$

$$\ln|y-2| = \frac{1}{5}x^5 + C$$

$$|y-2| = e^C e^{\frac{1}{5}x^5}$$

$$y-2 = Ke^{\frac{1}{5}x^5}, K = \pm e^C$$

$$-2 = Ke^0 = K$$

$$y = 2 - 2e^{\frac{1}{5}x^5}$$

9.

$$(a) \quad 2yy' = y + xy'$$

$$(2y - x)y' = y$$

$$y' = \frac{y}{2y - x}$$

$$(b) \quad \frac{y}{2y - x} = \frac{1}{2}$$

$$2y = 2y - x$$

$$x = 0$$

$$y = \pm\sqrt{2}$$

$$(0, \sqrt{2}), (0, -\sqrt{2})$$

$$(c) \quad \frac{y}{2y - x} = 0$$

$$y = 0$$

The curve has no horizontal tangent since $0^2 \neq 2 + x \cdot 0$ for any x .

$$(d) \quad \text{When } y = 3, \quad 3^2 = 2 + 3x \text{ so } x = \frac{7}{3}.$$

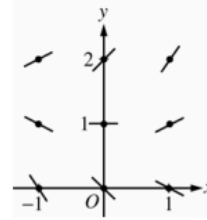
$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{y}{2y - x} \cdot \frac{dx}{dt}$$

$$\text{At } t = 5, \quad 6 = \frac{3}{6 - \frac{7}{3}} \cdot \frac{dx}{dt} = \frac{9}{11} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} \Big|_{t=5} = \frac{22}{3}$$

10.

(a)



$$(b) \quad \frac{d^2y}{dx^2} = \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2}x + y - \frac{1}{2}$$

Solution curves will be concave up on the half-plane above the line $y = -\frac{1}{2}x + \frac{1}{2}$.

$$(c) \quad \frac{dy}{dx} \Big|_{(0,1)} = 0 + 1 - 1 = 0 \text{ and } \frac{d^2y}{dx^2} \Big|_{(0,1)} = 0 + 1 - \frac{1}{2} > 0$$

Thus, f has a relative minimum at $(0, 1)$.

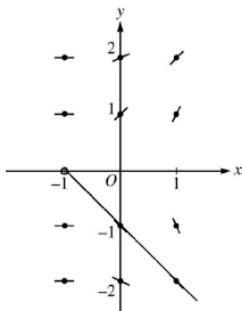
(d) Substituting $y = mx + b$ into the differential equation:

$$m = \frac{1}{2}x + (mx + b) - 1 = \left(m + \frac{1}{2}\right)x + (b - 1)$$

$$\text{Then } 0 = m + \frac{1}{2} \text{ and } m = b - 1: m = -\frac{1}{2} \text{ and } b = \frac{1}{2}.$$

11.

(a)



$$(b) \quad -1 = \frac{x+1}{y} \Rightarrow y = -x - 1$$

$$\frac{dy}{dx} = -1 \text{ for all } (x, y) \text{ with } y = -x - 1 \text{ and } y \neq 0$$

$$(c) \quad \int y \, dy = \int (x+1) \, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + x + C$$

$$\frac{(-2)^2}{2} = \frac{0^2}{2} + 0 + C \Rightarrow C = 2$$

$$y^2 = x^2 + 2x + 4$$

Since the solution goes through $(0, -2)$, y must be negative. Therefore $y = -\sqrt{x^2 + 2x + 4}$.

12.

$$(a) \quad \text{Area} = \int_0^4 \sqrt{x} \, dx + \frac{1}{2} \cdot 2 \cdot 2 = \frac{2}{3}x^{3/2} \Big|_{x=0}^{x=4} + 2 = \frac{22}{3}$$

$$(b) \quad y = \sqrt{x} \Rightarrow x = y^2$$

$$y = 6 - x \Rightarrow x = 6 - y$$

$$\text{Width} = (6 - y) - y^2$$

$$\text{Volume} = \int_0^2 2y(6 - y - y^2) \, dy$$

$$(c) \quad g'(x) = -1$$

Thus a line perpendicular to the graph of g has slope 1.

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = 1 \Rightarrow x = \frac{1}{4}$$

The point P has coordinates $\left(\frac{1}{4}, \frac{1}{2}\right)$.

13.

$$(a) \int_0^2 (6 - 4\ln(3-x)) dx = 6.816 \text{ or } 6.817$$

$$(b) \pi \int_0^2 ((8 - 4\ln(3-x))^2 - (8-6)^2) dx \\ = 168.179 \text{ or } 168.180$$

$$(c) \int_0^2 (6 - 4\ln(3-x))^2 dx = 26.266 \text{ or } 26.267$$

14.

$$(a) \text{ Area} = \int_0^4 \left(\sqrt{x} - \frac{x}{2} \right) dx = \frac{2}{3} x^{3/2} - \frac{x^2}{4} \Big|_{x=0}^{x=4} = \frac{4}{3}$$

$$(b) \text{ Volume} = \int_0^4 \left(\sqrt{x} - \frac{x}{2} \right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4} \right) dx \\ = \frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12} \Big|_{x=0}^{x=4} = \frac{8}{15}$$

$$(c) \text{ Volume} = \pi \int_0^4 \left(\left(2 - \frac{x}{2} \right)^2 - (2 - \sqrt{x})^2 \right) dx$$

15.

The graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$ intersect at the points $(0, 0)$ and $(9, 3)$.

$$(a) \int_0^9 \left(\sqrt{x} - \frac{x}{3} \right) dx = 4.5$$

OR

$$\int_0^3 (3y - y^2) dy = 4.5$$

$$(b) \pi \int_0^3 \left((3y+1)^2 - (y^2+1)^2 \right) dy \\ = \frac{207\pi}{5} = 130.061 \text{ or } 130.062$$

$$(c) \int_0^3 (3y - y^2)^2 dy = 8.1$$

16.

$$e^{2x-x^2} = 2 \text{ when } x = 0.446057, 1.553943 \\ \text{Let } P = 0.446057 \text{ and } Q = 1.553943$$

$$(a) \text{ Area of } R = \int_P^Q (e^{2x-x^2} - 2) dx = 0.514$$

$$(b) e^{2x-x^2} = 1 \text{ when } x = 0, 2$$

$$\text{Area of } S = \int_0^2 (e^{2x-x^2} - 1) dx - \text{Area of } R \\ = 2.06016 - \text{Area of } R = 1.546$$

OR

$$\int_0^P (e^{2x-x^2} - 1) dx + (Q-P) \cdot 1 + \int_Q^2 (e^{2x-x^2} - 1) dx \\ = 0.219064 + 1.107886 + 0.219064 = 1.546$$

$$(c) \text{ Volume} = \pi \int_P^Q \left((e^{2x-x^2} - 1)^2 - (2-1)^2 \right) dx$$

17.

For $x < 0$, $f(x) = 0$ when $x = -1.37312$.Let $P = -1.37312$.

(a) Area of $R = \int_P^0 f(x) dx = 2.903$

(b) Volume $= \pi \int_P^0 ((f(x) + 2)^2 - 4) dx = 59.361$

(c) The equation of the tangent line ℓ is $y = 3 - \frac{1}{2}x$.

The graph of f and line ℓ intersect at $A = 3.38987$.

Area of $S = \int_0^A \left(\left(3 - \frac{1}{2}x \right) - f(x) \right) dx$

18.

The graphs of f and g intersect in the first quadrant at $(S, T) = (1.13569, 1.76446)$.

(a) Area $= \int_0^S (f(x) - g(x)) dx$
 $= \int_0^S (1 + \sin(2x) - e^{x/2}) dx$
 $= 0.429$

(b) Volume $= \pi \int_0^S ((f(x))^2 - (g(x))^2) dx$
 $= \pi \int_0^S ((1 + \sin(2x))^2 - (e^{x/2})^2) dx$
 $= 4.266$ or 4.267

(c) Volume $= \int_0^S \frac{\pi}{2} \left(\frac{f(x) - g(x)}{2} \right)^2 dx$
 $= \int_0^S \frac{\pi}{2} \left(\frac{1 + \sin(2x) - e^{x/2}}{2} \right)^2 dx$
 $= 0.077$ or 0.078

19.

(a) $f'(x) = 8x - 3x^2$; $f'(3) = 24 - 27 = -3$

$f(3) = 36 - 27 = 9$

Tangent line at $x = 3$ is

$y = -3(x - 3) + 9 = -3x + 18$,

which is the equation of line ℓ .

(b) $f(x) = 0$ at $x = 4$

The line intersects the x -axis at $x = 6$.

Area $= \frac{1}{2}(3)(9) - \int_3^4 (4x^2 - x^3) dx$
 $= 7.916$ or 7.917

OR

Area $= \int_3^4 ((18 - 3x) - (4x^2 - x^3)) dx$
 $+ \frac{1}{2}(2)(18 - 12)$
 $= 7.916$ or 7.917

(c) Volume $= \pi \int_0^4 (4x^2 - x^3)^2 dx$
 $= 156.038\pi$ or 490.208

20.

Region R

$\frac{x^3}{1+x^2} = 4 - 2x$ at $x = 1.487664 = A$

(a) Area $= \int_0^A \left(4 - 2x - \frac{x^3}{1+x^2} \right) dx$
 $= 3.214$ or 3.215

(b) Volume $= \pi \int_0^A \left((4 - 2x)^2 - \left(\frac{x^3}{1+x^2} \right)^2 \right) dx$
 $= 31.884$ or 31.885 or 10.149π

(c) Volume $= \int_0^A \left(4 - 2x - \frac{x^3}{1+x^2} \right)^2 dx$
 $= 8.997$

21.

$$\begin{aligned}
 \text{(a) Area} &= \int_0^1 \frac{x^2}{x^2+1} dx \\
 &= \int_0^1 1 - \frac{1}{x^2+1} dx \\
 &= x - \arctan x \Big|_0^1 \\
 &= 1 - \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) Volume} &= 2\pi \int_0^1 x \left(\frac{x^2}{x^2+1} \right) dx \\
 &= 2\pi \int_0^1 x - \frac{x}{x^2+1} dx \\
 &= 2\pi \left(\frac{x^2}{2} - \frac{1}{2} \ln|x^2+1| \right) \Big|_0^1 \\
 &= \pi(1 - \ln 2)
 \end{aligned}$$

22.

a)

$$\begin{aligned}
 \text{(a) } A &= \int_0^1 e^x - (x-1)^2 dx \\
 &= \int_0^1 e^x - x^2 + 2x - 1 dx \\
 &= e^x \Big|_0^1 - \frac{1}{3}(x-1)^3 \Big|_0^1 \\
 &= (e-1) - \frac{1}{3} = e - \frac{4}{3}
 \end{aligned}$$

b)

$$\pi \int_0^1 (e^{2x} - (x-1)^4) dx = \pi \left(\frac{e^2}{2} - \frac{7}{10} \right)$$