AP Calculus AB: Ch 3 Free Response Packet

1. No Calculator

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

2. No Calculator

Let f be the function given by $f(x) = 3x^4 + x^3 - 21x^2$.

(a) Write an equation of the line tangent to the graph of f at the point $(2, -28)$.

(b) Find the absolute minimum value of f. Show the analysis that leads to your conclusion.

(c) Find the x-coordinate of each point of inflection on the graph of f. Show the analysis that leads to your conclusion.

3. No Calculator

Let f be the function given by $f(x) = x^3 - 7x + 6$.

(a) Find the zeros of f.

(b) Write an equation of the line tangent to the graph of f at $x = -1$.

(c) Find the number c that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[1, 3]$.

4. No Calculator

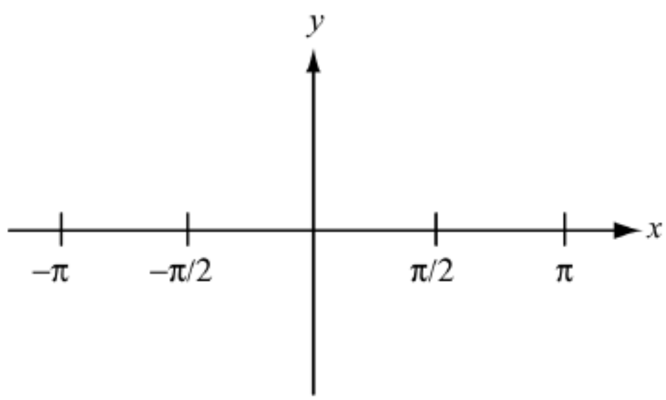
Given the function f defined by $f(x) = \cos x - \cos^2 x$ for $-\pi \leq x \leq \pi$.

(a) Find the x-intercepts of the graph of f.

(b) Find the x- and y-coordinates of all relative maximum points of f. Justify your answer.

(c) Find the intervals on which the graph of f is increasing.

(d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.



5. No Calculator

Let f be the function defined for $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ by $f(x) = x + \sin^2 x$.

(a) Find all values of x for which $f'(x) = 1$.

(b) Find the x-coordinates of all minimum points of f. Justify your answer.

(c) Find the x-coordinates of all inflection points of f. Justify your answer.

6. No Calculator

Let f be the function defined by $y = f(x) = x^3 + ax^2 + bx + c$ and having the following properties.

(i) The graph of f has a point of inflection at $(0, -2)$.
(ii) The average (mean) value of $f(x)$ on the closed interval $[0, 2]$ is -3.

(a) Determine the values of a, b, and c.

(b) Determine the value of x that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[0, 3]$.

7. No Calculator

Let f be the function defined by $f(x) = 3x^5 - 5x^3 + 2$.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave upward?

(c) Write the equation of each horizontal tangent line to the graph of f.

8. No Calculator

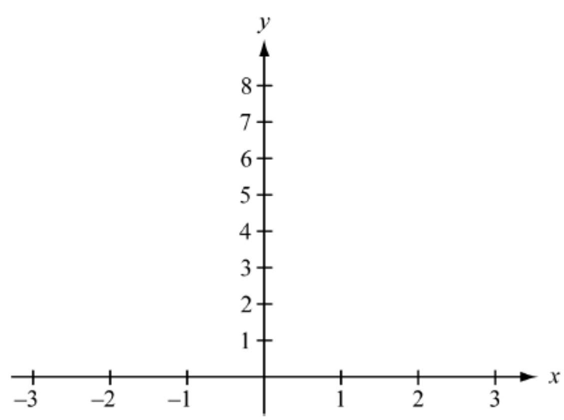
A function f is continuous on the closed interval $[-3, 3]$ such that $f(-3) = 4$ and $f(3) = 1$. The functions f' and f'' have the properties given in the table below.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Positive	Fails to exist	Negative	0	Negative
$f''(x)$	Positive	Fails to exist	Positive	0	Negative

(a) What are the x-coordinates of all absolute maximum and absolute minimum points of f on the interval $[-3, 3]$? Justify your answer.

(b) What are the x-coordinates of all points of inflection of f on the interval $[-3, 3]$? Justify your answer.

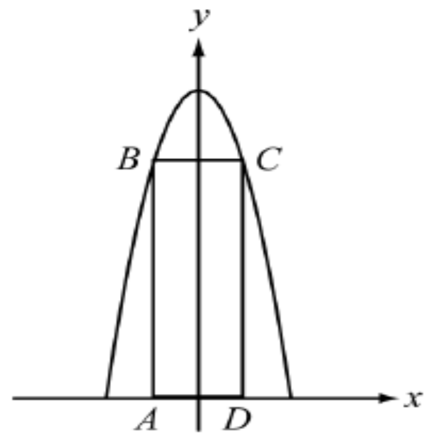
(c) On the axes provided, sketch a graph that satisfies the given properties of f.



9. No Calculator

Find the maximum volume of a box that can be made by cutting out squares from the corners of an 8-inch by 15-inch rectangular sheet of cardboard and folding up the sides. Justify your answer.

10. No Calculator

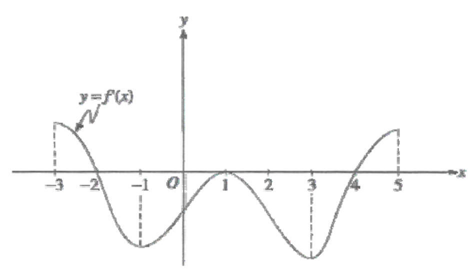


A rectangle $ABCD$ with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of $y = -4x^2 + 4$ and the x-axis as shown in the figure above.

(a) Find the x- and y-coordinates of C so that the area of rectangle $ABCD$ is a maximum.

(b) The point C moves along the curve with its x-coordinate increasing at the constant rate of 2 units per second. Find the rate of change of the area of rectangle $ABCD$ when $x = \frac{1}{2}$.

11. No Calculator



Note: This is the graph of the derivative of f, not the graph of f.

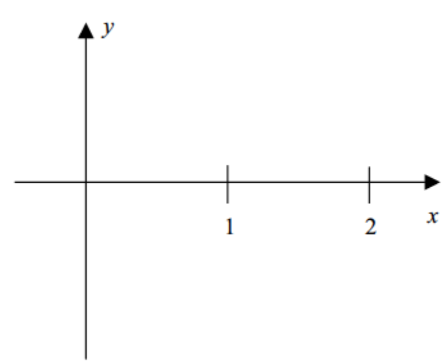
The figure above shows the graph of f', the derivative of a function f. The domai is the set of all real numbers x such that $-3 < x < 5$.

(a) For what values of x does f have a relative maximum? Why?

(b) For what values of x does f have a relative minimum? Why?

(c) On what intervals is the graph of f concave upward? Use f' to justify your answer.

(d) Suppose that $f(1) = 0$. In the xy-plane provided, draw a sketch that shows the general shape of the graph of the function f on the open interval $0 < x < 2$.



12. No Calculator

A particle moves along a line so that at any time t its position is given by $x(t) = 2\pi t + \cos 2\pi t$.

(a) Find the velocity at time t.

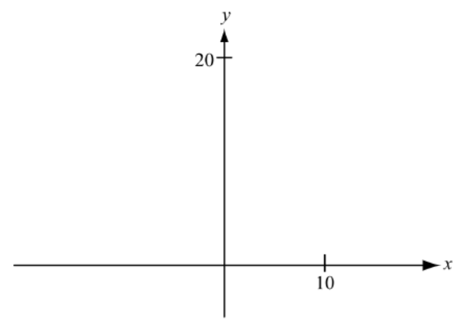
(b) Find the acceleration at time t.

(c) What are all values of t, $0 \le t \le 3$, for which the particle is at rest?

(d) What is the maximum velocity?

13. No Calculator

Let f be the function defined by $f(x) = 12x^{\frac{2}{3}} - 4x$.



(a) Find the intervals on which f is increasing.

(b) Find the x- and y-coordinates of all relative maximum points.

(c) Find the x- and y-coordinates of all relative minimum points.

(d) Find the intervals on which f is concave downward.

(e) Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.

14.

Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$. The graph of $y = P(x)$ is symmetric with respect to the y-axis, has a relative maximum at $(0,1)$, and an absolute minimum at $(q, -3)$.

(a) Determine the values of a, b, c, and d, and using these values write an expression for $P(x)$.

(b) Find all possible values of q.

15.

Let g and h be any two twice-differentiable functions that are defined for all real numbers and that satisfy the following properties for all x:

(i) $(g(x))^2 + (h(x))^2 = 1$

(ii) $g'(x) = (h(x))^2$

(iii) $h(x) > 0$

(iv) $g(0) = 0$

(a) Justify that $h'(x) = -g(x)h(x)$ for all x.

(b) Justify that h has a relative maximum at $x = 0$.

(c) Justify that the graph of g has a point of inflection at $x = 0$.

16.

A particle moves along the x-axis so that at any time $t > 0$ its velocity is given by $v(t) = t\ln t - t$. At time $t = 1$, the position of the particle is $x(1) = 6$.

(a) Write an expression for the acceleration of the particle.

(b) For what values of t is the particle moving to the right?

(c) What is the minimum velocity of the particle? Show the analysis that leads to your conclusion.

17.

Given the function f defined by $f(x) = \ln(x^2 - 9)$.

(a) Describe the symmetry of the graph of f.

(b) Find the domain of f.

(c) Find all values of x such that $f(x) = 0$.

(d) Write a formula for $f^{-1}(x)$, the inverse function of f, for $x > 3$.

18.

A particle moves along the x-axis in such a way that at time $t > 0$ its position coordinate is $x = \sin(e^t)$.

(a) Find the velocity and acceleration of the particle at time t.

(b) At what time does the particle first have zero velocity?

(c) What is the acceleration of the particle at the time determined in part (b)?

19.

A function f is defined by $f(x) = xe^{-2x}$ with domain $0 \le x \le 10$.

(a) Find all values of x for which the graph of f is increasing and all values of x for which the graph is decreasing.

(b) Give the x- and y-coordinates of all absolute maximum and minimum points on the graph of f. Justify your answers.

20.

A particle moves on the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = 2te^{-t}$.

(a) Find the acceleration of the particle at $t = 0$.

(b) Find the velocity of the particle when its acceleration is 0.

ANSWERS

1.

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$

$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$

Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$

Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

2.

(a) $f'(x) = 12x^3 + 3x^2 - 42x$

$f'(2) = 24$

$y + 28 = 24(x - 2)$

or $y = 24x - 76$

(b) $12x^3 + 3x^2 - 42x = 0$

$3x(4x^2 + x - 14) = 0$

$3x(4x - 7)(x + 2) = 0$

$x = 0, x = \frac{7}{4}, x = -2$

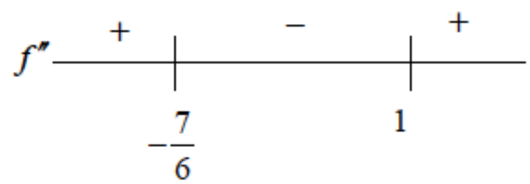
Absolute min is -44

(c) $f''(x) = 36x^2 + 6x - 42$

$= 6(6x^2 + x - 7)$

$= 6(6x + 7)(x - 1)$

Zeros at $x = -\frac{7}{6}$, $x = 1$



The x coordinates of the points of inflection are $x = -\frac{7}{6}$ and $x = 1$

3.

(a) $f(x) = x^3 - 7x + 6$

$= (x-1)(x-2)(x+3)$

$x = 1,\ x = 2,\ x = -3$

(b) $f'(x) = 3x^2 - 7$

$f'(-1) = -4,\ f(-1) = 12$

$y - 12 = -4(x+1)$

or

$4x + y = 8$

or

$y = -4x + 8$

(c) $\dfrac{f(3) - f(1)}{3-1} = \dfrac{12-0}{2} = 6$

$3c^2 - 7 = f'(c) = 6$

$c = \sqrt{\dfrac{13}{3}}$

4.

(a) $f(x) = \cos x \cdot (1 - \cos x)$

Either $\cos x = 0$ or $1 - \cos x = 0$, so the x-intercepts are $x = -\dfrac{\pi}{2}$, $x = \dfrac{\pi}{2}$, and $x = 0$.

(b) $f'(x) = -\sin x + 2\sin x \cos x$

$0 = \sin x \cdot (-1 + 2\cos x)$

Either $\sin x = 0$ or $\cos x = \dfrac{1}{2}$, so the candidates are $x = \pm\pi$, $x = 0$, and $x = \pm\dfrac{\pi}{3}$.

The relative maximum points are at $\left(\pm\dfrac{\pi}{3}, \dfrac{1}{4}\right)$.

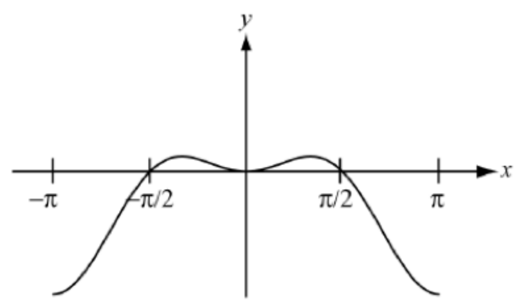
Justification:

(i) $f''(x) = -\cos x + 2\cos 2x$

$f''(\pm\pi) = 3 \Rightarrow$ relative minimum

$f''(0) = 1 \Rightarrow$ relative minimum

$f''\left(\pm\dfrac{\pi}{3}\right) = -\dfrac{3}{2} \Rightarrow$ relative maximum



(c) Graph of f increases on the intervals $-\pi < x < -\dfrac{\pi}{3}$ and $0 < x < \dfrac{\pi}{3}$.

5.

(a) $f'(x) = 1 + 2\sin x \cos x = 1 + \sin 2x$

$1 = 1 + \sin 2x$

$x = \frac{\pi}{2}$ is the only solution in the interval $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$.

(b) $f'(x) = 1 + \sin 2x = 0$, so $x = \frac{3\pi}{4}$

The minimum occurs at the critical point or at the endpoints.

critical point: $f\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4} + \frac{1}{2} = 2.856$

endpoints: $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \frac{1}{4} = 0.774$

$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + \frac{1}{4} = 2.868$

Therefore the minimum is at $x = \frac{\pi}{6}$.

(c) $f''(x) = 2\cos 2x$

$2\cos 2x = 0$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

f''	$\pi/6$		$\pi/4$		$3\pi/4$		$5\pi/6$
		+		–		+	

Therefore the inflection points occur at $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ since this is where f'' changes sign from positive to negative and from negative to positive, respectively.

6.

(a) $-2 = f(0) = c$

$f'(x) = 3x^2 + 2ax + b$

$f''(x) = 6x + 2a$

$0 = f''(0) = 2a$, so $a = 0$

$f(x) = x^3 + bx - 2$

$$-3 = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\left(\frac{x^4}{4} + \frac{bx^2}{2} - 2x\right)\Bigg|_0^2 = \frac{1}{2}(4 + 2b - 4) = b$$

So $a = 0$, $b = -3$, and $c = -2$.

7.

(a) $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$

Sign of f'		-1		0		1	
	+		–		–		+

Answer: f is increasing on the intervals $(-\infty, -1]$ and $[1, \infty)$

(b) $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$

sign of f''		$-\frac{1}{\sqrt{2}}$		0		$\frac{1}{\sqrt{2}}$	
	–		+		–		+

Answer: f is concave upward on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and on $\left(\frac{1}{\sqrt{2}}, \infty\right)$

(b) By the Mean Value Theorem, there is an x satisfying $0 < x < 3$ such that

$$f'(x) = \frac{f(3) - f(0)}{3 - 0}$$

$$3x^2 - 3 = \frac{16 - (-2)}{3} = 6$$

$$x^2 = 3 \Rightarrow x = \sqrt{3}$$

(c) $f'(x) = 0$ when $x = -1, 0, 1$

$$x = -1 \Rightarrow f(x) = 4;\ y = 4$$

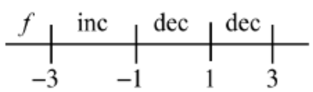
$$f(0) = 2;\ y = 2$$

$$f(1) = 0;\ y = 0$$

8.

(a) The absolute maximum occurs at $x = -1$ because f is increasing on the interval $[-3, -1]$ and decreasing on the interval $[-1, 3]$.

or

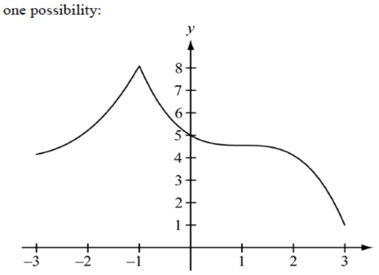


The absolute minimum must occur at $x = 1$ (the other critical point) or at an endpoint. However, f is decreasing on the interval $[-1, 3]$. Therefore the absolute minimum is at an endpoint. Since $f(-3) = 4 > 1 = f(3)$, the absolute minimum is at $x = 3$.

(b) There is an inflection point at $x = 1$ because:

the graph of f changes from concave up to concave down at $x = 1$

(c) This is one possibility:



9.

$V(x) = x(8-2x)(15-2x) = 4x^3 - 46x^2 + 120x$

$V'(x) = 12x^2 - 92x + 120$

$3x^2 - 23x + 30 = (3x-5)(x-6) = 0$

$x = \frac{5}{3},\ x = 6$

Since we must have $0 \le x \le 4$, we pick $x = \frac{5}{3}$.

$$V_{\max} = \frac{5}{3}\left(8 - \frac{10}{3}\right)\left(15 - \frac{10}{3}\right) = \frac{5}{3}\cdot\frac{14}{3}\cdot\frac{35}{3} = \frac{2450}{27} = 90\frac{20}{27} \approx 90.7$$

$V''\left(\frac{5}{3}\right) < 0$ and so there is a relative maximum at $x = \frac{5}{3}$. There is only one critic point in the domain $0 \le x \le 4$, so there is an absolute maximum at $x = \frac{5}{3}$.

10.

(a) $A(x) = 2x(-4x^2 + 4) = 8(x - x^3)$

$\frac{dA}{dx} = 8(1 - 3x^2)$

$\frac{dA}{dx} = 0$ when $x = \frac{1}{\sqrt{3}}$

The maximum area occurs when $x = \frac{1}{\sqrt{3}}$ and $y = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$.

(b) $A(x) = 8(x - x^3)$

$\frac{dA}{dt} = 8(1 - 3x^2)\frac{dx}{dt}$

When $x = \frac{1}{2}$ and $\frac{dx}{dt} = 2$, $\frac{dA}{dt} = 8\left(1 - 3\cdot\frac{1}{4}\right)2 = 4$.

11.

(a) $x = -2$

$f'(x)$ changes from positive to negative at $x = -2$

or

f is increasing to the left of $x = -2$ and decreasing to the right of $x = -2$

(b) $x = 4$

$f'(x)$ changes from negative to positive at $x = 4$

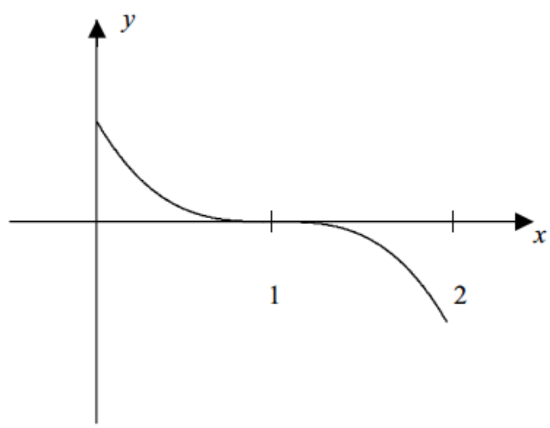
or

f is decreasing to the left of $x = 4$ and increasing to the right of $x = 4$

(c) $(-1,1)$ and $(3,5)$

f' is increasing on these intervals.

(d)



12.

(a) $v(t) = 2\pi - 2\pi\sin 2\pi t = 2\pi(1 - \sin 2\pi t)$

(b) $a(t) = -4\pi^2\cos 2\pi t$

(c) $v(t) = 2\pi(1 - \sin 2\pi t) = 0$

$\sin 2\pi t = 1$

The particle is at rest for $t = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}$.

(d) $a(t) = -4\pi^2\cos 2\pi t = 0$

$t = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots$

The maximum velocity is $v\left(\frac{3}{4}\right) = 4\pi$.

or

Since $\sin 2\pi t = -1$ is the minimum of $\sin 2\pi t$, the maximum of $v(t)$ is $2\pi(1 - (-1)) = 4\pi$.

13.

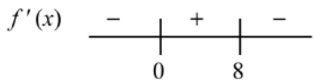
(a) $f(x)=12x^{2/3}-4x;\ f'(x)=8x^{-1/3}-4$

$(8x^{-1/3}-4)>0,\ x>0 \Rightarrow x<8$

$(8x^{-1/3}-4)>0,\ x<0 \Rightarrow$ no x satisfies this

or

Critical numbers: $x=8$, $x=0$



Therefore f is increasing on the interval $0<x<8$.

(b) 2nd Derivative Test: $f''(x)=-\frac{8}{3}x^{-4/3}$

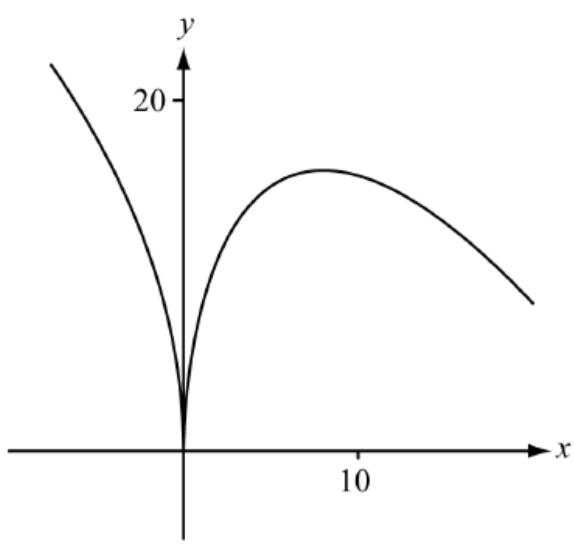
$f''(8)<0 \Rightarrow$ relative maximum at $(8,16)$

(c) The 2nd Derivative Test cannot be used at $x=0$ where the second derivative is undefined. Since $f'(x)<0$ for x just less than 0, and $f'(x)>0$ for x just greater than 0, there is a relative minimum at $(0,0)$.

(d) $f''(x)=-\frac{8}{3}x^{-4/3}<0$ if $x\neq 0$

The graph of f is concave down on $(-\infty,0)$ and $(0,+\infty)$.

(e)



14.

a) $P(x)=x^4-4x^2+1$

b) $q=\pm\sqrt{2}$

15.

(a) $2g(x)g'(x)+2h(x)h'(x)=0$ from differentiating both sides of (i)

$g(x)(h(x))^2+h(x)h'(x)=0$ from (ii)

Since $h(x)\neq 0$ by (iii), we must have $g(x)h(x)+h'(x)=0$.

Therefore $h'(x)=-g(x)h(x)$

(b) Since $g(0)=0$, $h'(0)=-g(0)h(0)=0$.

$h''(x)=-g(x)h'(x)-g'(x)h(x)$

$h''(0)=-g(0)h'(0)-g'(0)h(0)=0-h(0)^2h(0)=-h(0)^3<0$ since $h(x)>0$ for all x.

Therefore h has a relative maximum at $x=0$.

Alternatively, since $g'(x)=(h(x))^2$, g is an increasing function. Since $g(0)=0$, we must have that $g(x)<0$ for $x<0$ and $g(x)>0$ for $x>0$. Now $h'(x)=-g(x)h(x)$ and $h(x)>0$. Therefore $h'(x)>0$ for $x<0$ and $h'(x)<0$ for $x>0$. Hence h has a relative maximum at $x=0$.

(c) $g''(x)=2h(x)h'(x)=-2g(x)(h(x))^2$. Therefore $g''(x)$ changes sign at $x=0$ because $h(x)\neq 0$ and $g(x)$ changes sign at $x=0$ (see part (b)). This implies that the graph of g has a point of inflection at $x=0$.

16.

(a) $a(t)=v'(t)=\ln t+t\cdot\frac{1}{t}-1=\ln t$

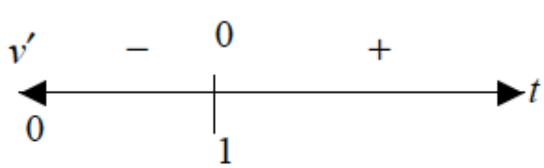
(b) $v(t)=t\ln t-t>0$

$t(\ln t-1)>0$

$t>e$

(c) $v'(t)=\ln t=0$

$t=1$



minimum velocity is $v(1)=-1$

17.

(a) $f(-x)=\ln((-x)^2-9)=\ln(x^2-9)=f(x)$
Therefore the graph of f is symmetric with respect to the y-axis.

(b) Since we need $x^2-9>0$, the domain of f is the set $\{x \mid x<-3 \text{ or } x>3\}$

(c) $f(x)=0$ when $x^2-9=1$. This happens for $x=\pm\sqrt{10}$.

(d) Method 1:

$f(x)=\ln(x^2-9) \Rightarrow x^2-9=e^{f(x)}=e^y$
Since $x>3$, $x=\sqrt{e^y+9}$.

Hence $f^{-1}(x)=\sqrt{e^x+9}$.

18.

(a) $x=\sin(e^t)$

$$v=\frac{dx}{dt}=e^t\cos(e^t)$$

$$a=\frac{dv}{dt}=e^t(\cos(e^t)-e^t\sin(e^t))$$

(b) $v(t)=0$ when $\cos(e^t)=0$. Hence $e^t=\frac{\pi}{2}$ gives the first time when the velocity is zero, and so $t=\ln\frac{\pi}{2}$.

(c) $$a\left(\ln\frac{\pi}{2}\right)=\frac{\pi}{2}\left(\cos\frac{\pi}{2}-\frac{\pi}{2}\sin\frac{\pi}{2}\right)=-\frac{\pi^2}{4}$$

19.

a. increasing on [0, ½]
decreasing on [½ , 10]

b. Abs max: $\left(\frac{1}{2},\frac{1}{2e}\right)$

abs min. (0, 0)

20.

a. -4

b. $\frac{-2}{e^2}$