

AP Calculus AB: Ch 3 Free Response Packet

1. No Calculator

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x . The function h is given by $h(x) = f(g(x)) - 6$.

- (a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.
- (b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

2. No Calculator

Let f be the function given by $f(x) = 3x^4 + x^3 - 21x^2$.

- (a) Write an equation of the line tangent to the graph of f at the point $(2, -28)$.
- (b) Find the absolute minimum value of f . Show the analysis that leads to your conclusion.
- (c) Find the x -coordinate of each point of inflection on the graph of f . Show the analysis that leads to your conclusion.

3. No Calculator

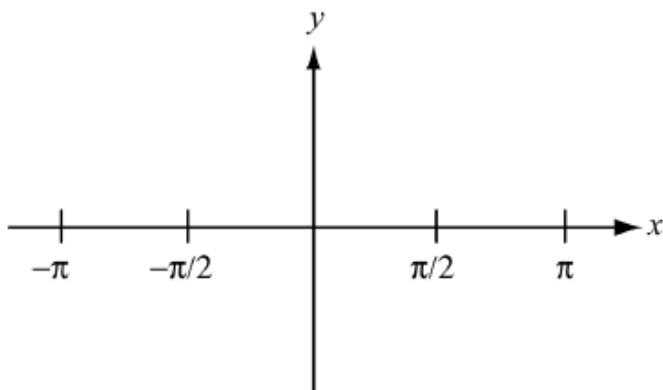
Let f be the function given by $f(x) = x^3 - 7x + 6$.

- (a) Find the zeros of f .
- (b) Write an equation of the line tangent to the graph of f at $x = -1$.
- (c) Find the number c that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[1, 3]$.

4. No Calculator

Given the function f defined by $f(x) = \cos x - \cos^2 x$ for $-\pi \leq x \leq \pi$.

- (a) Find the x -intercepts of the graph of f .
- (b) Find the x - and y -coordinates of all relative maximum points of f . Justify your answer.
- (c) Find the intervals on which the graph of f is increasing.
- (d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.



5. No Calculator

Let f be the function defined for $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ by $f(x) = x + \sin^2 x$.

- (a) Find all values of x for which $f'(x) = 1$.
- (b) Find the x -coordinates of all minimum points of f . Justify your answer.
- (c) Find the x -coordinates of all inflection points of f . Justify your answer.

6. No Calculator

Let f be the function defined by $y = f(x) = x^3 + ax^2 + bx + c$ and having the following properties.

- (i) The graph of f has a point of inflection at $(0, -2)$.
 - (ii) The average (mean) value of $f(x)$ on the closed interval $[0, 2]$ is -3 .
- (a) Determine the values of a , b , and c .
 - (b) Determine the value of x that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[0, 3]$.

7. No Calculator

Let f be the function defined by $f(x) = 3x^5 - 5x^3 + 2$.

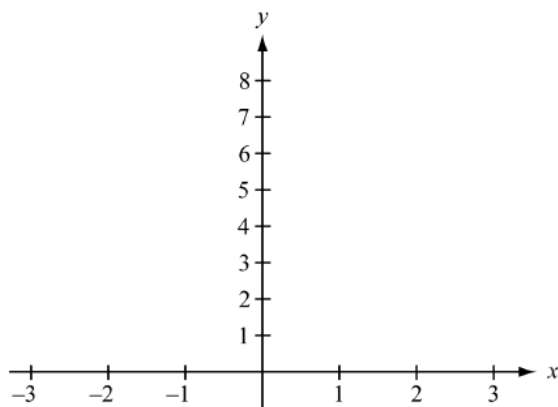
- (a) On what intervals is f increasing?
- (b) On what intervals is the graph of f concave upward?
- (c) Write the equation of each horizontal tangent line to the graph of f .

8. No Calculator

A function f is continuous on the closed interval $[-3, 3]$ such that $f(-3) = 4$ and $f(3) = 1$. The functions f' and f'' have the properties given in the table below.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Positive	Fails to exist	Negative	0	Negative
$f''(x)$	Positive	Fails to exist	Positive	0	Negative

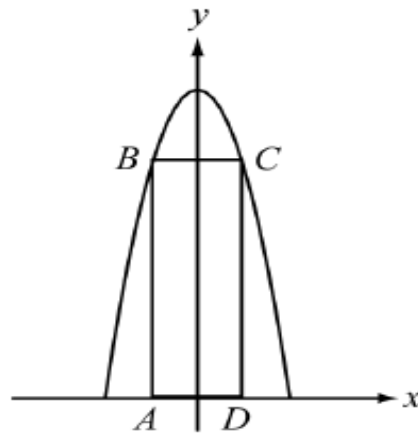
- (a) What are the x -coordinates of all absolute maximum and absolute minimum points of f on the interval $[-3, 3]$? Justify your answer.
- (b) What are the x -coordinates of all points of inflection of f on the interval $[-3, 3]$? Justify your answer.
- (c) On the axes provided, sketch a graph that satisfies the given properties of f .



9. No Calculator

Find the maximum volume of a box that can be made by cutting out squares from the corners of an 8-inch by 15-inch rectangular sheet of cardboard and folding up the sides. Justify your answer.

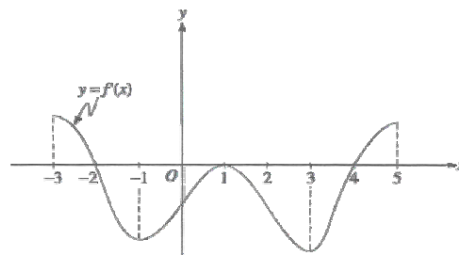
10. No Calculator



A rectangle $ABCD$ with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of $y = -4x^2 + 4$ and the x -axis as shown in the figure above.

- Find the x - and y -coordinates of C so that the area of rectangle $ABCD$ is a maximum.
- The point C moves along the curve with its x -coordinate increasing at the constant rate of 2 units per second. Find the rate of change of the area of rectangle $ABCD$ when $x = \frac{1}{2}$.

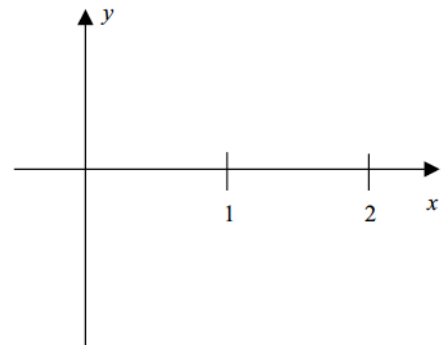
11. No Calculator



Note: This is the graph of the derivative of f , not the graph of f .

The figure above shows the graph of f' , the derivative of a function f . The domain is the set of all real numbers x such that $-3 < x < 5$.

- For what values of x does f have a relative maximum? Why?
- For what values of x does f have a relative minimum? Why?
- On what intervals is the graph of f concave upward? Use f' to justify your answer.
- Suppose that $f(1) = 0$. In the xy -plane provided, draw a sketch that shows the general shape of the graph of the function f on the open interval $0 < x < 2$.



12. No Calculator

A particle moves along a line so that at any time t its position is given by

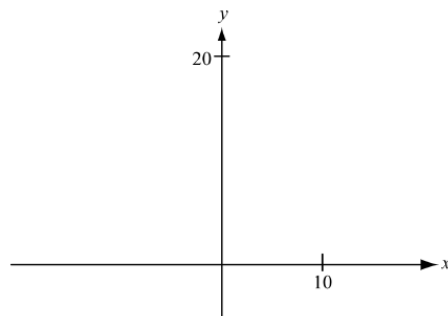
$$x(t) = 2\pi t + \cos 2\pi t.$$

- Find the velocity at time t .
- Find the acceleration at time t .
- What are all values of t , $0 \leq t \leq 3$, for which the particle is at rest?
- What is the maximum velocity?

13. No Calculator

Let f be the function defined by $f(x) = 12x^{\frac{2}{3}} - 4x$.

- Find the intervals on which f is increasing.
- Find the x - and y -coordinates of all relative maximum points.
- Find the x - and y -coordinates of all relative minimum points.
- Find the intervals on which f is concave downward.
- Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.



14.

Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$. The graph of $y = P(x)$ is symmetric with respect to the y -axis, has a relative maximum at $(0, 1)$, and an absolute minimum at $(q, -3)$.

- Determine the values of a , b , c , and d , and using these values write an expression for $P(x)$.
- Find all possible values of q .

15.

Let g and h be any two twice-differentiable functions that are defined for all real numbers and that satisfy the following properties for all x :

- $(g(x))^2 + (h(x))^2 = 1$
- $g'(x) = (h(x))^2$
- $h(x) > 0$
- $g(0) = 0$

- Justify that $h'(x) = -g(x)h(x)$ for all x .
- Justify that h has a relative maximum at $x = 0$.
- Justify that the graph of g has a point of inflection at $x = 0$.

16.

A particle moves along the x -axis so that at any time $t > 0$ its velocity is given by $v(t) = t \ln t - t$. At time $t = 1$, the position of the particle is $x(1) = 6$.

- (a) Write an expression for the acceleration of the particle.
- (b) For what values of t is the particle moving to the right?
- (c) What is the minimum velocity of the particle? Show the analysis that leads to your conclusion.

17.

Given the function f defined by $f(x) = \ln(x^2 - 9)$.

- (a) Describe the symmetry of the graph of f .
- (b) Find the domain of f .
- (c) Find all values of x such that $f(x) = 0$.
- (d) Write a formula for $f^{-1}(x)$, the inverse function of f , for $x > 3$.

18.

A particle moves along the x -axis in such a way that at time $t > 0$ its position coordinate is $x = \sin(e^t)$.

- (a) Find the velocity and acceleration of the particle at time t .
- (b) At what time does the particle first have zero velocity?
- (c) What is the acceleration of the particle at the time determined in part (b)?

19.

A function f is defined by $f(x) = xe^{-2x}$ with domain $0 \leq x \leq 10$.

- (a) Find all values of x for which the graph of f is increasing and all values of x for which the graph is decreasing.
- (b) Give the x - and y -coordinates of all absolute maximum and minimum points on the graph of f . Justify your answers.

20.

A particle moves on the x -axis so that its position at any time $t \geq 0$ is given by $x(t) = 2te^{-t}$.

- (a) Find the acceleration of the particle at $t = 0$.
- (b) Find the velocity of the particle when its acceleration is 0.

ANSWERS

1.

$$(a) \quad h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$$

$$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$$

Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r , $1 < r < 3$, such that $h(r) = -5$.

$$(b) \quad \frac{h(3) - h(1)}{3 - 1} = \frac{-7 - 3}{3 - 1} = -5$$

Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c , $1 < c < 3$, such that $h'(c) = -5$.

2.

$$(a) \quad f'(x) = 12x^3 + 3x^2 - 42x$$

$$f'(2) = 24$$

$$y + 28 = 24(x - 2)$$

$$\text{or } y = 24x - 76$$

$$(b) \quad 12x^3 + 3x^2 - 42x = 0$$

$$3x(4x^2 + x - 14) = 0$$

$$3x(4x - 7)(x + 2) = 0$$

$$x = 0, x = \frac{7}{4}, x = -2$$

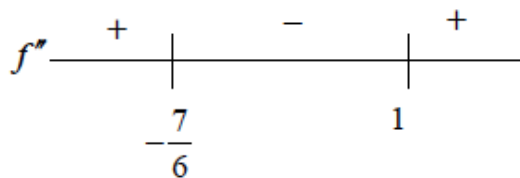
Absolute min is -44

$$(c) \quad f''(x) = 36x^2 + 6x - 42$$

$$= 6(6x^2 + x - 7)$$

$$= 6(6x + 7)(x - 1)$$

$$\text{Zeros at } x = -\frac{7}{6}, x = 1$$



The x coordinates of the points of inflection are $x = -\frac{7}{6}$ and $x = 1$

3.

$$\begin{aligned} \text{(a)} \quad f(x) &= x^3 - 7x + 6 \\ &= (x-1)(x-2)(x+3) \\ x &= 1, x = 2, x = -3 \end{aligned}$$

$$\text{(b)} \quad f'(x) = 3x^2 - 7$$

$$f'(-1) = -4, f(-1) = 12$$

$$y - 12 = -4(x + 1)$$

or

$$4x + y = 8$$

or

$$y = -4x + 8$$

$$\text{(c)} \quad \frac{f(3) - f(1)}{3 - 1} = \frac{12 - 0}{2} = 6$$

$$3c^2 - 7 = f'(c) = 6$$

$$c = \sqrt{\frac{13}{3}}$$

4.

$$\text{(a)} \quad f(x) = \cos x \cdot (1 - \cos x)$$

Either $\cos x = 0$ or $1 - \cos x = 0$, so the x -intercepts are $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = 0$.

$$\text{(b)} \quad f'(x) = -\sin x + 2 \sin x \cos x$$

$$0 = \sin x \cdot (-1 + 2 \cos x)$$

Either $\sin x = 0$ or $\cos x = \frac{1}{2}$, so the candidates are $x = \pm\pi$, $x = 0$, and $x = \pm\frac{\pi}{3}$.

The relative maximum points are at $\left(\pm\frac{\pi}{3}, \frac{1}{4}\right)$.

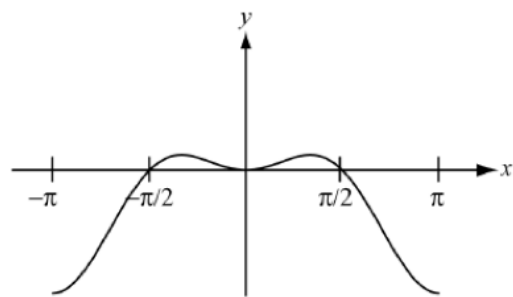
Justification:

$$\text{(i)} \quad f''(x) = -\cos x + 2 \cos 2x$$

$$f''(\pm\pi) = 3 \Rightarrow \text{relative minimum}$$

$$f''(0) = 1 \Rightarrow \text{relative minimum}$$

$$f''\left(\pm\frac{\pi}{3}\right) = -\frac{3}{2} \Rightarrow \text{relative maximum}$$



(c) Graph of f increases on the intervals $-\pi < x < -\frac{\pi}{3}$ and $0 < x < \frac{\pi}{3}$.

5.

(a) $f'(x) = 1 + 2 \sin x \cos x = 1 + \sin 2x$

$$1 = 1 + \sin 2x$$

$$x = \frac{\pi}{2} \text{ is the only solution in the interval } \frac{\pi}{6} \leq x \leq \frac{5\pi}{6}.$$

(b) $f'(x) = 1 + \sin 2x = 0$, so $x = \frac{3\pi}{4}$

The minimum occurs at the critical point or at the endpoints.

critical point: $f\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4} + \frac{1}{2} = 2.856$

endpoints: $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \frac{1}{4} = 0.774$

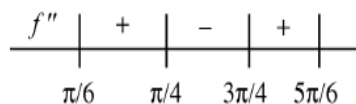
$$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + \frac{1}{4} = 2.868$$

Therefore the minimum is at $x = \frac{\pi}{6}$.

(c) $f''(x) = 2 \cos 2x$

$$2 \cos 2x = 0$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$



Therefore the inflection points occur at $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ since this is where f'' changes sign from positive to negative and from negative to positive, respectively.

6.

(a) $-2 = f(0) = c$

$$f'(x) = 3x^2 + 2ax + b$$

$$f''(x) = 6x + 2a$$

$$0 = f''(0) = 2a, \text{ so } a = 0$$

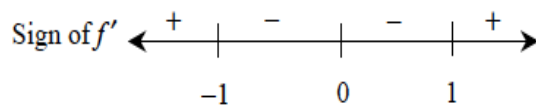
$$f(x) = x^3 + bx - 2$$

$$-3 = \frac{1}{2-0} \int_0^2 f(x) dx = \frac{1}{2} \left(\frac{x^4}{4} + \frac{bx^2}{2} - 2x \right) \Big|_0^2 = \frac{1}{2} (4 + 2b - 4) = b$$

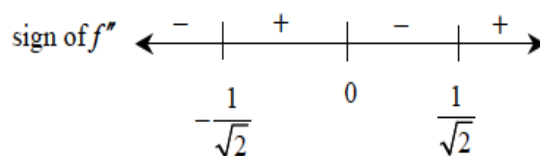
So $a = 0$, $b = -3$, and $c = -2$.

7.

(a) $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$

Answer: f is increasing on the intervals $(-\infty, -1]$ and $[1, \infty)$

(b) $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$

Answer: f is concave upward on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and on $\left(\frac{1}{\sqrt{2}}, \infty\right)$

(b) By the Mean Value Theorem, there is an x satisfying $0 < x < 3$ such that (c) $f'(x) = 0$ when $x = -1, 0, 1$

$$f'(x) = \frac{f(3) - f(0)}{3 - 0}$$

$$3x^2 - 3 = \frac{16 - (-2)}{3} = 6$$

$$x^2 = 3 \Rightarrow x = \sqrt{3}$$

$$x = -1 \Rightarrow f(x) = 4; y = 4$$

$$f(0) = 2; y = 2$$

$$f(1) = 0; y = 0$$

8.

(a) The absolute maximum occurs at $x = -1$ because f is increasing on the interval $[-3, -1]$ and decreasing on the interval $[-1, 3]$.

or

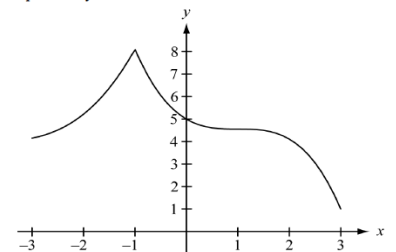
f	inc	dec	dec
-3	-1	1	3

The absolute minimum must occur at $x = 1$ (the other critical point) or at an endpoint. However, f is decreasing on the interval $[-1, 3]$. Therefore the absolute minimum is at an endpoint. Since $f(-3) = 4 > 1 = f(3)$, the absolute minimum is at $x = 3$.

(b) There is an inflection point at $x = 1$ because:

the graph of f changes from concave up to concave down at $x = 1$

(c) This is one possibility:



9.

$$V(x) = x(8-2x)(15-2x) = 4x^3 - 46x^2 + 120x$$

$$V'(x) = 12x^2 - 92x + 120$$

$$3x^2 - 23x + 30 = (3x-5)(x-6) = 0$$

$$x = \frac{5}{3}, x = 6$$

Since we must have $0 \leq x \leq 4$, we pick $x = \frac{5}{3}$.

$$V_{\max} = \frac{5}{3} \left(8 - \frac{10}{3} \right) \left(15 - \frac{10}{3} \right) = \frac{5}{3} \cdot \frac{14}{3} \cdot \frac{35}{3} = \frac{2450}{27} = 90 \frac{20}{27} \approx 90.7$$

$V''\left(\frac{5}{3}\right) < 0$ and so there is a relative maximum at $x = \frac{5}{3}$. There is only one critical point in the domain $0 \leq x \leq 4$, so there is an absolute maximum at $x = \frac{5}{3}$.

10.

$$(a) A(x) = 2x(-4x^2 + 4) = 8(x - x^3)$$

$$\frac{dA}{dx} = 8(1 - 3x^2)$$

$$\frac{dA}{dx} = 0 \text{ when } x = \frac{1}{\sqrt{3}}$$

The maximum area occurs when $x = \frac{1}{\sqrt{3}}$ and $y = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$.

$$(b) A(x) = 8(x - x^3)$$

$$\frac{dA}{dt} = 8(1 - 3x^2) \frac{dx}{dt}$$

$$\text{When } x = \frac{1}{2} \text{ and } \frac{dx}{dt} = 2, \frac{dA}{dt} = 8\left(1 - 3 \cdot \frac{1}{4}\right) 2 = 4.$$

11.

$$(a) x = -2$$

$f'(x)$ changes from positive to negative at $x = -2$

or

f is increasing to the left of $x = -2$ and decreasing to the right of $x = -2$

$$(b) x = 4$$

$f'(x)$ changes from negative to positive at $x = 4$

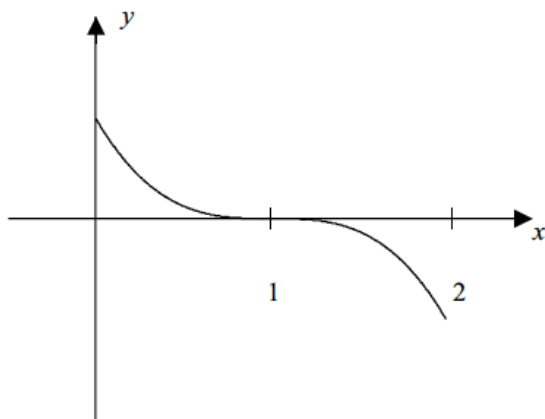
or

f is decreasing to the left of $x = 4$ and increasing to the right of $x = 4$

$$(c) (-1, 1) \text{ and } (3, 5)$$

f' is increasing on these intervals.

(d)



12.

$$(a) v(t) = 2\pi - 2\pi \sin 2\pi t = 2\pi(1 - \sin 2\pi t)$$

$$(b) a(t) = -4\pi^2 \cos 2\pi t$$

$$(c) v(t) = 2\pi(1 - \sin 2\pi t) = 0$$

$$\sin 2\pi t = 1$$

The particle is at rest for $t = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}$.

$$(d) a(t) = -4\pi^2 \cos 2\pi t = 0$$

$$t = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \dots$$

The maximum velocity is $v\left(\frac{3}{4}\right) = 4\pi$.

or

Since $\sin 2\pi t = -1$ is the minimum of $\sin 2\pi t$, the maximum of $v(t)$ is $2\pi(1 - (-1)) = 4\pi$.

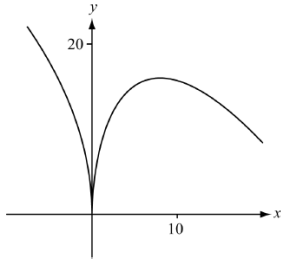
13.

(c) The 2nd Derivative Test cannot be used at $x = 0$ where the second derivative is undefined. Since $f'(x) < 0$ for x just less than 0, and $f'(x) > 0$ for x just greater than 0, there is a relative minimum at $(0, 0)$.

(d) $f''(x) = -\frac{8}{3}x^{-4/3} < 0$ if $x \neq 0$

The graph of f is concave down on $(-\infty, 0)$ and $(0, +\infty)$.

(e)



14.

a) $P(x) = x^4 - 4x^2 + 1$

b) $q = \pm\sqrt{2}$

15.

(a) $2g(x)g'(x) + 2h(x)h'(x) = 0$ from differentiating both sides of (i)

$$g(x)(h(x))^2 + h(x)h'(x) = 0 \text{ from (ii)}$$

Since $h(x) \neq 0$ by (iii), we must have $g(x)h(x) + h'(x) = 0$.

$$\text{Therefore } h'(x) = -g(x)h(x)$$

(b) Since $g(0) = 0$, $h'(0) = -g(0)h(0) = 0$.

$$h''(x) = -g(x)h'(x) - g'(x)h(x)$$

$$h''(0) = -g(0)h'(0) - g'(0)h(0) = 0 - h(0)^2 h'(0) = -h(0)^3 < 0 \text{ since } h(x) > 0 \text{ for all } x.$$

Therefore h has a relative maximum at $x = 0$.

Alternatively, since $g'(x) = (h(x))^2$, g is an increasing function. Since $g(0) = 0$, we must have that $g(x) < 0$ for $x < 0$ and $g(x) > 0$ for $x > 0$. Now $h'(x) = -g(x)h(x)$ and $h(x) > 0$. Therefore $h'(x) > 0$ for $x < 0$ and $h'(x) < 0$ for $x > 0$. Hence h has a relative maximum at $x = 0$.

(c) $g''(x) = 2h(x)h'(x) = -2g(x)(h(x))^2$. Therefore $g''(x)$ changes sign at $x = 0$ because $h(x) \neq 0$ and $g(x)$ changes sign at $x = 0$ (see part (b)). This implies that the graph of g has a point of inflection at $x = 0$.

16.

(a) $f(x)$ is defined everywhere except $x = 0$

(b) The zeros are $\pm e$

(c) $y = 2x - 4$

17.

(a) $f(-x) = \ln((-x)^2 - 9) = \ln(x^2 - 9) = f(x)$

Therefore the graph of f is symmetric with respect to the y -axis.

(b) Since we need $x^2 - 9 > 0$, the domain of f is the set $\{x \mid x < -3 \text{ or } x > 3\}$

(c) $f(x) = 0$ when $x^2 - 9 = 1$. This happens for $x = \pm\sqrt{10}$.

(d) Method 1:

$$f(x) = \ln(x^2 - 9) \Rightarrow x^2 - 9 = e^{f(x)} = e^y$$

$$\text{Since } x > 3, x = \sqrt{e^y + 9}.$$

$$\text{Hence } f^{-1}(x) = \sqrt{e^x + 9}.$$

18.

(a) $x = \sin(e^t)$

$$v = \frac{dx}{dt} = e^t \cos(e^t)$$

$$a = \frac{dv}{dt} = e^t(\cos(e^t) - e^t \sin(e^t))$$

(b) $v(t) = 0$ when $\cos(e^t) = 0$. Hence $e^t = \frac{\pi}{2}$ gives the first time when the velocity is

$$\text{zero, and so } t = \ln \frac{\pi}{2}.$$

(c) $a\left(\ln \frac{\pi}{2}\right) = \frac{\pi}{2} \left(\cos \frac{\pi}{2} - \frac{\pi}{2} \sin \frac{\pi}{2} \right) = -\frac{\pi^2}{4}$

19.

a. increasing on $[0, \frac{1}{2}]$
decreasing on $[\frac{1}{2}, 10]$

b. Abs max: $\left(\frac{1}{2}, \frac{1}{2e}\right)$

abs min. $(0, 0)$

20.

a. -4

b. $\frac{-2}{e^2}$