AP Calculus AB: Ch 3 Free Response Packet

1. No Calculator

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

2. No Calculator

Let f be the function given by $f(x) = 3x^4 + x^3 - 21x^2$.

(a) Write an equation of the line tangent to the graph of f at the point $(2, -28)$.

(b) Find the absolute minimum value of f. Show the analysis that leads to your conclusion.

(c) Find the x-coordinate of each point of inflection on the graph of f. Show the analysis that leads to your conclusion.

3. No Calculator

Let f be the function given by $f(x) = x^3 - 7x + 6$.

(a) Find the zeros of f.

(b) Write an equation of the line tangent to the graph of f at $x = -1$.

(c) Find the number c that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[1, 3]$.

4. No Calculator

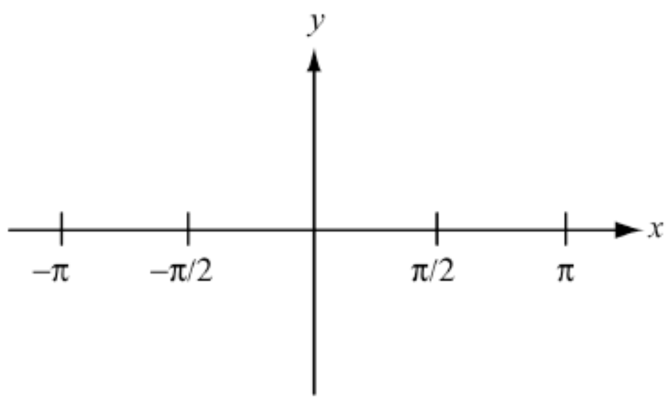
Given the function f defined by $f(x) = \cos x - \cos^2 x$ for $-\pi \le x \le \pi$.

(a) Find the x-intercepts of the graph of f.

(b) Find the x- and y-coordinates of all relative maximum points of f. Justify your answer.

(c) Find the intervals on which the graph of f is increasing.

(d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.



5. No Calculator

Let f be the function defined for $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$ by $f(x) = x + \sin^2 x$.

(a) Find all values of x for which $f'(x) = 1$.

(b) Find the x-coordinates of <u>all</u> minimum points of f. Justify your answer.

(c) Find the x-coordinates of <u>all</u> inflection points of f. Justify your answer.

6. No Calculator

Let f be the function defined by $y = f(x) = x^3 + ax^2 + bx + c$ and having the following properties.

(i) The graph of f has a point of inflection at $(0, -2)$.
(ii) The average (mean) value of $f(x)$ on the closed interval $[0, 2]$ is -3.

(a) Determine the values of a, b, and c.

(b) Determine the value of x that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[0, 3]$.

7. No Calculator

Let f be the function defined by $f(x) = 3x^5 - 5x^3 + 2$.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave upward?

(c) Write the equation of each horizontal tangent line to the graph of f.

8. No Calculator

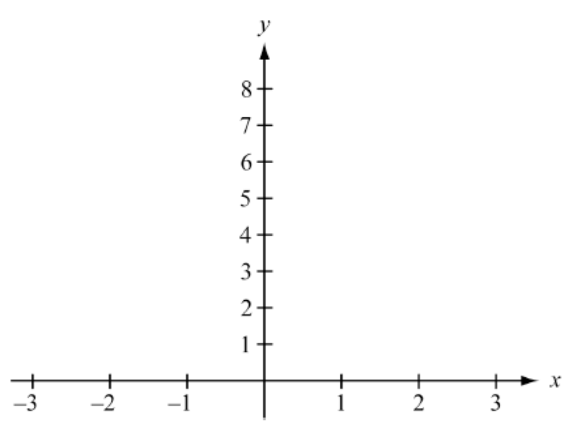
A function f is continuous on the closed interval $[-3, 3]$ such that $f(-3) = 4$ and $f(3) = 1$. The functions f' and f'' have the properties given in the table below.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Positive	Fails to exist	Negative	0	Negative
$f''(x)$	Positive	Fails to exist	Positive	0	Negative

(a) What are the x-coordinates of all absolute maximum and absolute minimum points of f on the interval $[-3, 3]$? Justify your answer.

(b) What are the x-coordinates of all points of inflection of f on the interval $[-3, 3]$? Justify your answer.

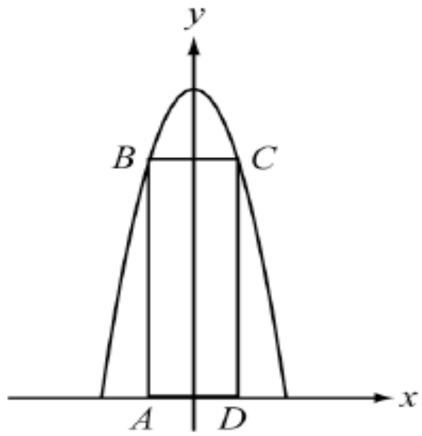
(c) On the axes provided, sketch a graph that satisfies the given properties of f.



9. No Calculator

Find the maximum volume of a box that can be made by cutting out squares from the corners of an 8-inch by 15-inch rectangular sheet of cardboard and folding up the sides. Justify your answer.

10. No Calculator

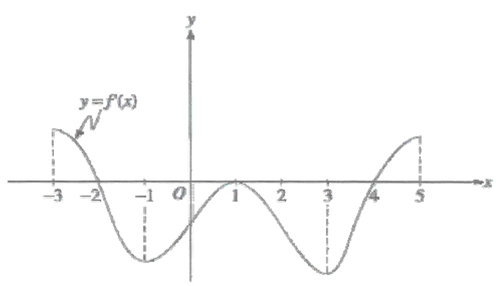


A rectangle $ABCD$ with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of $y = -4x^2 + 4$ and the x-axis as shown in the figure above.

(a) Find the x- and y-coordinates of C so that the area of rectangle $ABCD$ is a maximum.

(b) The point C moves along the curve with its x-coordinate increasing at the constant rate of 2 units per second. Find the rate of change of the area of rectangle $ABCD$ when $x = \frac{1}{2}$.

11. No Calculator



Note: This is the graph of the derivative of f, not the graph of f.

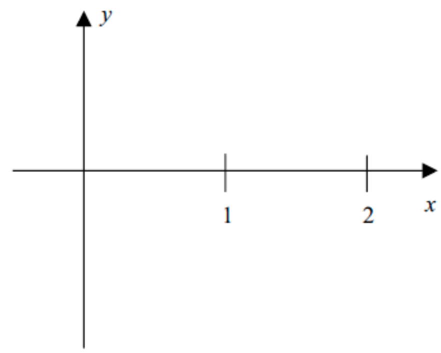
The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.

(a) For what values of x does f have a relative maximum? Why?

(b) For what values of x does f have a relative minimum? Why?

(c) On what intervals is the graph of f concave upward? Use f' to justify your answer.

(d) Suppose that $f(1) = 0$. In the xy-plane provided, draw a sketch that shows the general shape of the graph of the function f on the open interval $0 < x < 2$.



12. No Calculator

A particle moves along a line so that at any time t its position is given by $x(t) = 2\pi t + \cos 2\pi t$.

(a) Find the velocity at time t.

(b) Find the acceleration at time t.

(c) What are all values of t, $0 \le t \le 3$, for which the particle is at rest?

(d) What is the maximum velocity?

13. No Calculator

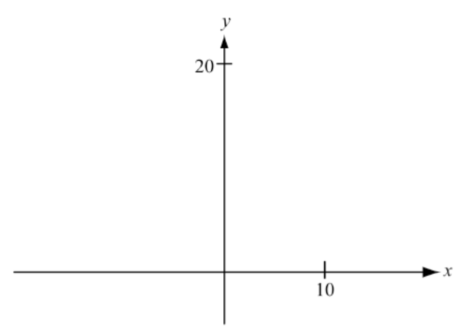
Let f be the function defined by $f(x) = 12x^{\frac{2}{3}} - 4x$.

(a) Find the intervals on which f is increasing.

(b) Find the x- and y-coordinates of all relative maximum points.

(c) Find the x- and y-coordinates of all relative minimum points.

(d) Find the intervals on which f is concave downward.



(e) Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.

ANSWERS

1.

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$

$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$

Since $h(3) < -5 < h(1)$ and h is continuous, by

Intermediate Value Theorem, there exists a valu

$1 < r < 3$, such that $h(r) = -5$.

(b) $\frac{h(3) - h(1)}{3 - 1} = \frac{-7 - 3}{3 - 1} = -5$

Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

2.

(a) $f'(x) = 12x^3 + 3x^2 - 42x$

$f'(2) = 24$

$y + 28 = 24(x - 2)$

or $y = 24x - 76$

(b) $12x^3 + 3x^2 - 42x = 0$

$3x(4x^2 + x - 14) = 0$

$3x(4x - 7)(x + 2) = 0$

$x = 0, x = \frac{7}{4}, x = -2$

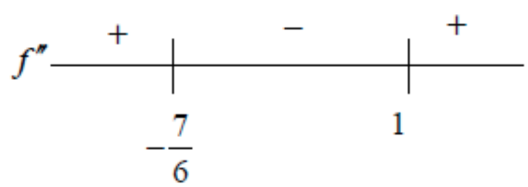
Absolute min is -44

(c) $f''(x) = 36x^2 + 6x - 42$

$= 6(6x^2 + x - 7)$

$= 6(6x + 7)(x - 1)$

Zeros at $x = -\frac{7}{6}$, $x = 1$



The x coordinates of the points of inflection are $x = -\frac{7}{6}$ and $x = 1$

3.

(a) $f(x) = x^3 - 7x + 6$

$= (x-1)(x-2)(x+3)$

$x = 1,\ x = 2,\ x = -3$

(b) $f'(x) = 3x^2 - 7$

$f'(-1) = -4,\ f(-1) = 12$

$y - 12 = -4(x+1)$

or

$4x + y = 8$

or

$y = -4x + 8$

(c) $\dfrac{f(3) - f(1)}{3-1} = \dfrac{12-0}{2} = 6$

$3c^2 - 7 = f'(c) = 6$

$c = \sqrt{\dfrac{13}{3}}$

4.

(a) $f(x) = \cos x \cdot (1 - \cos x)$

Either $\cos x = 0$ or $1 - \cos x = 0$, so the x-intercepts are $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = 0$.

(b) $f'(x) = -\sin x + 2\sin x \cos x$

$0 = \sin x \cdot (-1 + 2\cos x)$

Either $\sin x = 0$ or $\cos x = \frac{1}{2}$, so the candidates are $x = \pm\pi$, $x = 0$, and $x = \pm\frac{\pi}{3}$.

The relative maximum points are at $\left(\pm\frac{\pi}{3}, \frac{1}{4}\right)$.

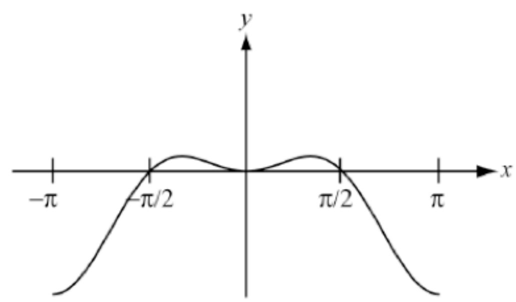
Justification:

(i) $f''(x) = -\cos x + 2\cos 2x$

$f''(\pm\pi) = 3 \Rightarrow$ relative minimum

$f''(0) = 1 \Rightarrow$ relative minimum

$f''\left(\pm\frac{\pi}{3}\right) = -\frac{3}{2} \Rightarrow$ relative maximum



(c) Graph of f increases on the intervals $-\pi < x < -\frac{\pi}{3}$ and $0 < x < \frac{\pi}{3}$.

5.

(a) $f'(x) = 1 + 2\sin x \cos x = 1 + \sin 2x$

$1 = 1 + \sin 2x$

$x = \frac{\pi}{2}$ is the only solution in the interval $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$.

(b) $f'(x) = 1 + \sin 2x = 0$, so $x = \frac{3\pi}{4}$

The minimum occurs at the critical point or at the endpoints.

critical point: $f\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4} + \frac{1}{2} = 2.856$

endpoints: $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \frac{1}{4} = 0.774$

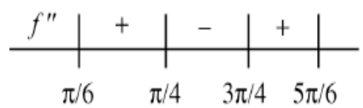
$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + \frac{1}{4} = 2.868$

Therefore the minimum is at $x = \frac{\pi}{6}$.

(c) $f''(x) = 2\cos 2x$

$2\cos 2x = 0$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$



Therefore the inflection points occur at $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ since this is where f'' changes sign from positive to negative and from negative to positive, respectively.

6.

(a) $-2 = f(0) = c$

$f'(x) = 3x^2 + 2ax + b$

$f''(x) = 6x + 2a$

$0 = f''(0) = 2a$, so $a = 0$

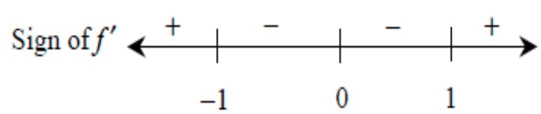
$f(x) = x^3 + bx - 2$

$-3 = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\left(\frac{x^4}{4} + \frac{bx^2}{2} - 2x\right)\Big|_0^2 = \frac{1}{2}(4 + 2b -$

So $a = 0$, $b = -3$, and $c = -2$.

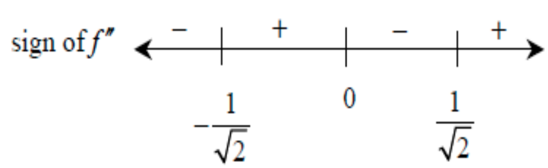
7.

(a) $f'(x) = 15x^4 - 15x^2 = 15x^2\left(x^2 - 1\right)$



Answer: f is increasing on the intervals $(-\infty, -1]$ and $[1, \infty)$

(b) $f''(x) = 60x^3 - 30x = 30x\left(2x^2 - 1\right)$



Answer: f is concave upward on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and on $\left(\frac{1}{\sqrt{2}}, \infty\right)$

(b) By the Mean Value Theorem, there is an x satisfying $0 < x < 3$ su

$$f'(x) = \frac{f(3) - f(0)}{3 - 0}$$

$$3x^2 - 3 = \frac{16 - (-2)}{3} = 6$$

$$x^2 = 3 \Rightarrow x = \sqrt{3}$$

(c) $f'(x) = 0$ when $x = -1, 0, 1$

$$x = -1 \Rightarrow f(x) = 4;\ y = 4$$

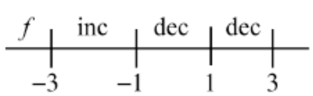
$$f(0) = 2;\ y = 2$$

$$f(1) = 0;\ y = 0$$

8.

(a) The absolute maximum occurs at $x = -1$ because f is increasing on the interval $[-3, -1]$ and decreasing on the interval $[-1, 3]$.

or

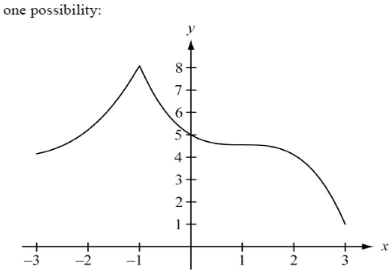


The absolute minimum must occur at $x = 1$ (the other critical point) or at an endpoint. However, f is decreasing on the interval $[-1, 3]$. Therefore the absolute minimum is at an endpoint. Since $f(-3) = 4 > 1 = f(3)$, the absolute minimum is at $x = 3$.

(b) There is an inflection point at $x = 1$ because:

the graph of f changes from concave up to concave down at $x = 1$

(c) This is one possibility:



9.

$V(x) = x(8-2x)(15-2x) = 4x^3 - 46x^2 + 120x$

$V'(x) = 12x^2 - 92x + 120$

$3x^2 - 23x + 30 = (3x-5)(x-6) = 0$

$x = \frac{5}{3},\ x = 6$

Since we must have $0 \le x \le 4$, we pick $x = \frac{5}{3}$.

$V_{max} = \frac{5}{3}\left(8 - \frac{10}{3}\right)\left(15 - \frac{10}{3}\right) = \frac{5}{3}\cdot\frac{14}{3}\cdot\frac{35}{3} = \frac{2450}{27} = 90\frac{20}{27}$

$V''\left(\frac{5}{3}\right) < 0$ and so there is a relative maximum at $x = \frac{5}{3}$. There is onl

point in the domain $0 \le x \le 4$, so there is an absolute maximum at $x =$

10.

(a) $A(x) = 2x(-4x^2 + 4) = 8(x - x^3)$

$\frac{dA}{dx} = 8(1 - 3x^2)$

$\frac{dA}{dx} = 0$ when $x = \frac{1}{\sqrt{3}}$

The maximum area occurs when $x = \frac{1}{\sqrt{3}}$ and $y = 4\left(1 - \frac{1}{3}\right) = \frac{8}{3}$.

(b) $A(x) = 8(x - x^3)$

$\frac{dA}{dt} = 8(1 - 3x^2)\frac{dx}{dt}$

When $x = \frac{1}{2}$ and $\frac{dx}{dt} = 2$, $\frac{dA}{dt} = 8\left(1 - 3\cdot\frac{1}{4}\right)2 = 4$.

11.

(a) $x = -2$

$f'(x)$ changes from positive to negative at $x = -2$

or

f is increasing to the left of $x = -2$ and decreasing to the right

(b) $x = 4$

$f'(x)$ changes from negative to positive at $x = 4$

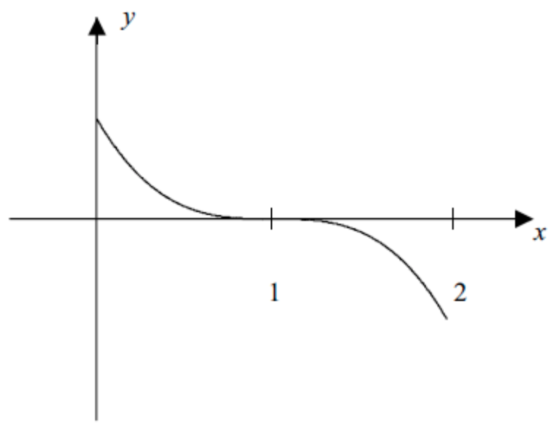
or

f is decreasing to the left of $x = 4$ and increasing to the right c

(c) $(-1,1)$ and $(3,5)$

f' is increasing on these intervals.

(d)



12.

(a) $v(t) = 2\pi - 2\pi \sin 2\pi t = 2\pi(1 - \sin 2\pi t)$

(b) $a(t) = -4\pi^2 \cos 2\pi t$

(c) $v(t) = 2\pi(1 - \sin 2\pi t) = 0$

$\sin 2\pi t = 1$

The particle is at rest for $t = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}$.

(d) $a(t) = -4\pi^2 \cos 2\pi t = 0$

$t = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots$

The maximum velocity is $v\left(\frac{3}{4}\right) = 4\pi$.

or

Since $\sin 2\pi t = -1$ is the minimum of $\sin 2\pi t$,

the maximum of $v(t)$ is $2\pi(1 - (-1)) = 4\pi$.

13.

(c) The 2nd Derivative Test cannot be used at $x = 0$ where the second derivative is undefined. Since $f'(x) < 0$ for x just less than 0, and $f'(x) > 0$ for x just greater than 0, there is a relative minimum at $(0, 0)$.

(d) $f''(x) = -\frac{8}{3}x^{-4/3} < 0$ if $x \neq 0$

The graph of f is concave down on $(-\infty, 0)$ and $(0, +\infty)$.

(e)