1.4 Continuity and One-Sided Limits

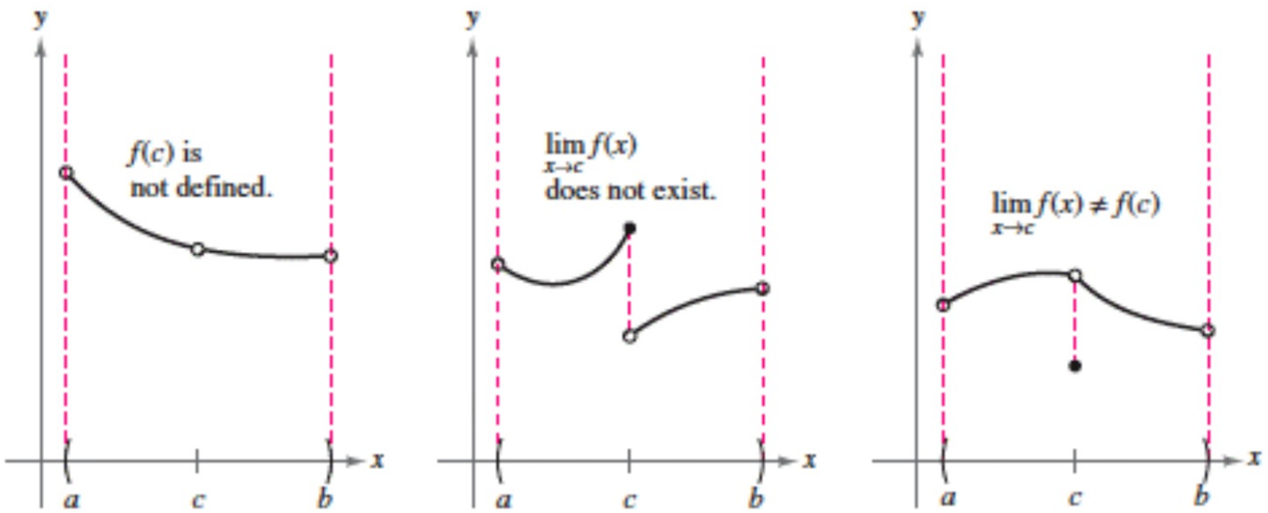
DEFINITION OF CONTINUITY

Continuity at a Point: A function f is **continuous at c** if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

$\lim_{x \to c} f(x) = f(c)$

Continuity on an Open Interval: A function is **continuous on an open interval (a, b)** if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**



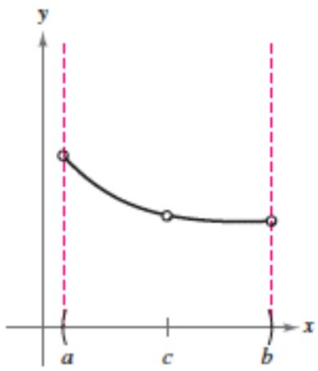
Three conditions exist for which the graph of f is not continuous at $x = c$.

In Figure 1.25, it appears that continuity at $x = c$ can be destroyed by any one of the following conditions.

1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but it is not equal to $f(c)$.

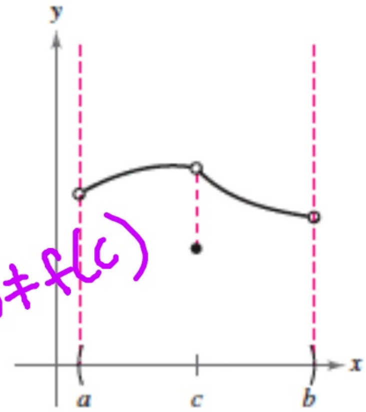
Removable Discontinuity:

If moving one point can 'fix' the discontinuity, this is called removable.

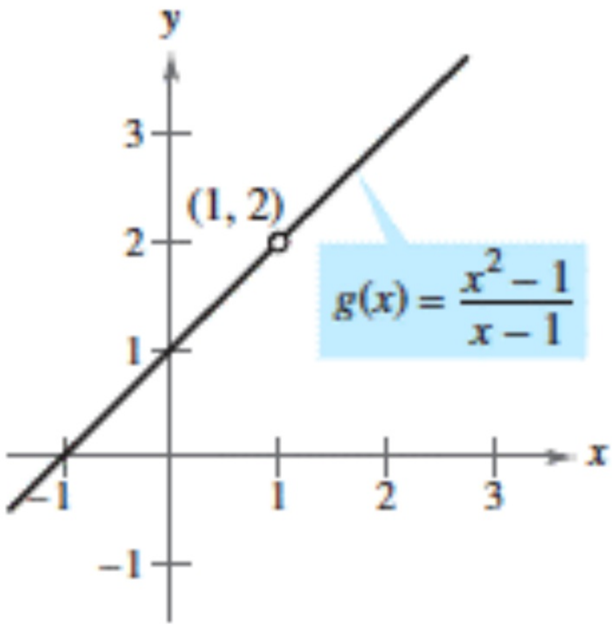


(a) Removable discontinuity

$\lim\limits_{x \to c} f(x)$ exists but $\lim\limits_{x \to c} f(x) \neq f(c)$



(c) Removable discontinuity



(b) Removable discontinuity at $x = 1$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

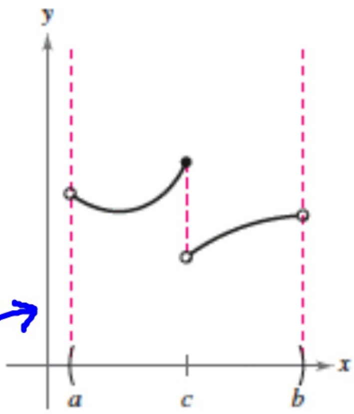
$$\lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

$$\lim_{x \to 1} (x+1)$$

$$2$$

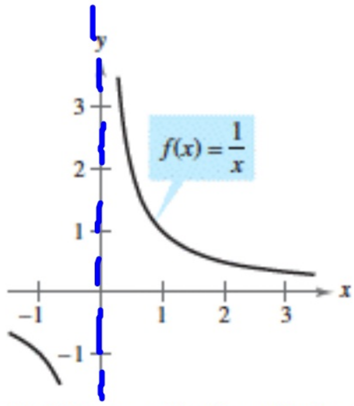
Nonremovable discontinuity:

If 'moving' more than one point is necessary to fix the discontinuity



(b) Nonremovable discontinuity

Jump discontinuity



(a) Nonremovable discontinuity at $x = 0$

THEOREM 1.11 PROPERTIES OF CONTINUITY

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$
2. Rational: $r(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$
3. Radical: $f(x) = \sqrt[n]{x}$
4. Trigonometric: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$

Discuss the continuity.

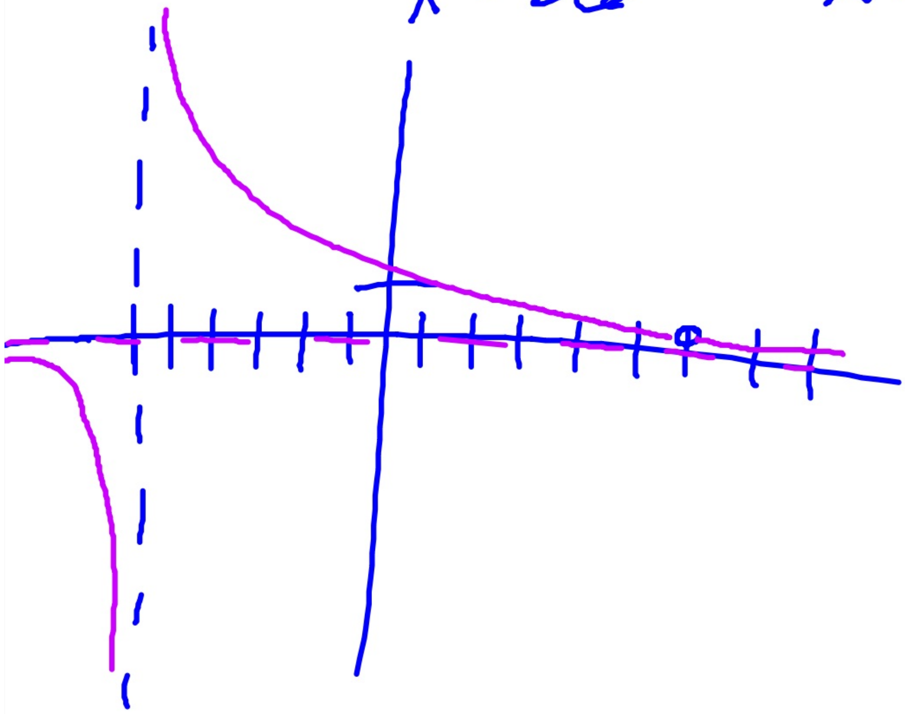
Ex 7

$$f(x) = \frac{x-6}{x^2-36} = \frac{\cancel{x-6}}{\cancel{(x-6)}(x+6)} = \frac{1}{x+6}$$

Removable discontinuity: $x=6$ (hole)

nonremovable discontinuity: $x=-6$ (VA)

$$f(x) = \frac{x-6}{x^2-36} = \frac{1}{x+6}$$



Functions that are continuous everywhere in their domain

$y = |x|$

$y = x + 1$

$y = -2x^2$

$y = x^3$

$y = \sqrt{x}$

$y = 2^x$

$y = \ln x$

$y = \sin x$

$y = \tan x$

$y = \cos x$

$y = \frac{1}{x}$

Functions that have domain all reals and are continuous everywhere.

$y = 5x - 1$

$y = |x|$

$y = 5x^2$

$y = x^3 - 7$

$y = \cos x$

$y = \sin x$

$y = \dfrac{x}{x^2 + 4}$

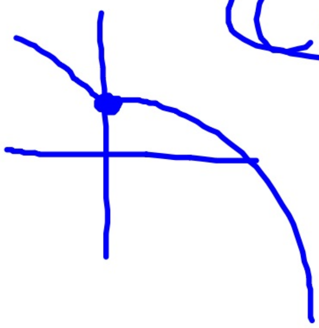
⑧ $g(x) = 7x - \cos 2x$

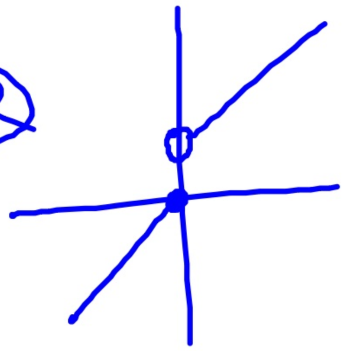
$7x$: Continuous everywhere

$\cos 2x$: Continuous everywhere

Continuous everywhere

$(-\infty, \infty)$

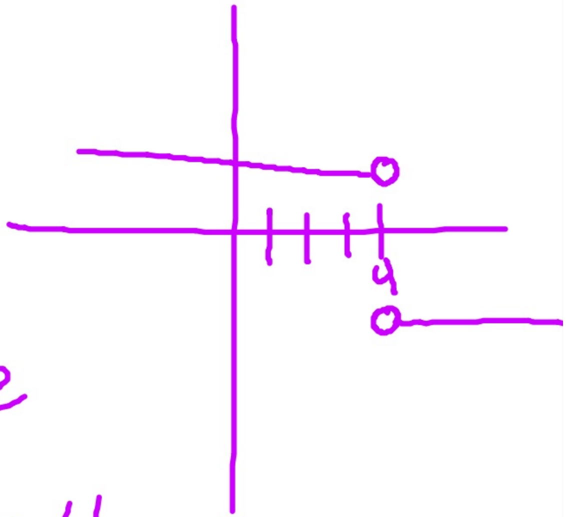




9 $g(x) = \dfrac{|4-x|}{4-x}$

x	y
0	1
1	1
2	1
3	1
5	-1
6	-1
7	-1

non Removable disc.
at $x=4$



10 $f(x) = \begin{cases} -2x+4, & x<1 \\ x^2, & x \geq 1 \end{cases}$ $\quad (1, 2)$ $\quad (1, 1)$

nonremovable
discontinuity
at $x=1$