# 1.3 Evaluating Limits Analytically

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using dividing out and rationalizing techniques.
- Evaluate a limit using the Squeeze Theorem.

**THEOREM 1.2 PROPERTIES OF LIMITS**

Let $b$ and $c$ be real numbers, let $n$ be a positive integer, and let $f$ and $g$ be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$
2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Strategy for finding limits

1. Direct Substitution
2. Algebraic techniques (factoring or rationalizing)
3. Special Cases

**THEOREM 1.9 TWO SPECIAL TRIGONOMETRIC LIMITS**

**1.** $\lim_{x \to 0} \frac{\sin x}{x} = 1$ **2.** $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

**Ex 1**

$$\lim_{x \to 2} \frac{\sqrt{x+2}}{x-4}$$

$$\frac{\sqrt{2+2}}{2-4}$$

$$\frac{2}{-2}$$

$$-1$$

**Ex 2**

$$\lim_{x \to 7} \sec\left(\frac{\pi x}{6}\right)$$

$$\sec\frac{7\pi}{6}$$

$$-\frac{2}{\sqrt{3}}$$

$$-\frac{2\sqrt{3}}{3}$$

Ex 3

$$\lim_{x \to -2} \tan\left(\frac{\pi x}{3}\right) = \tan\left(\frac{-2\pi}{3}\right)$$

$$\sqrt{3}$$

**Ex 4**

$$\lim_{x\to-1}\frac{2x^2-x-3}{x+1}$$

$$\lim_{x\to-1}\frac{\cancel{(x+1)}(2x-3)}{\cancel{x+1}}$$

$-5$

**Ex 5**

$$\lim_{x\to-1}\frac{x^3+1}{x+1}$$

$$\lim_{x\to-1}\frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{x+1}}$$

$(-1)^2-(-1)+1$

$3$

**Ex 6**

$$\lim_{x\to 3} \frac{(\sqrt{x+1}-2)}{x-3} \cdot \frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)}$$

$$\lim_{x\to 3} \frac{\cancel{x+1-4}}{\cancel{(x-3)}(\sqrt{x+1}+2)}$$

$$\lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$$

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \cdot \frac{(\sqrt{x+1} + 1)}{(\sqrt{x+1} + 1)}$$

$$\lim_{x \to 0} \frac{x+1-1}{x(\sqrt{x+1} + 1)}$$

$$\lim_{x \to 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{2}$$

**Ex 7**

$$\lim_{x\to 0} \frac{[1/(x+4)] - (1/4)}{x}$$

$$\lim_{x\to 0} \frac{\frac{1}{x+4}\cdot\frac{4}{4} - \frac{1}{4}\cdot\frac{(x+4)}{(x+4)}}{\frac{x}{1}\cdot 4(x+4)}$$

$$\lim_{x\to 0} \frac{4-(x+4)}{4x(x+4)}$$

$$\lim_{x\to 0} \frac{-\cancel{x}}{4\cancel{x}(x+4)}$$

$$\lim_{x\to 0} \frac{-1}{4(x+4)}$$

$$-\frac{1}{16}$$

**Ex 7**

$$\lim_{x\to 0} \frac{[1/(x+4)] - (1/4)}{x}$$

$$\lim_{x\to 0} \frac{\frac{1\cdot 4}{x+4} - \frac{1(x+4)}{4}}{x} = \lim_{x\to 0} \frac{\frac{4-(x+4)}{4(x+4)}}{x}$$

$$= \lim_{x\to 0} \frac{\frac{-x}{4(x+4)}}{\frac{x}{1}} = \lim_{x\to 0} \frac{-\cancel{x}}{4(x+4)} \cdot \frac{1}{\cancel{x}} = -\frac{1}{16}$$