## SERIES AND ERROR

The AP Calculus BC course description includes two kinds of error bounds:

- Alternating series with error bound
- Lagrange error bound for Taylor polynomials

Both types of error have been tested on the AP test. We will look at alternating series first.

An alternating series is a series whose terms are alternately positive and negative.
Examples: $\quad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{n}$

$$
-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots=\sum_{n=1}^{\infty}(-1)^{n} \frac{1}{n!}
$$

In general, just knowing that $\lim _{n \rightarrow \infty} a_{n}=0$ tells us very little about the convergence of the series $\sum_{n=1}^{\infty} a_{n}$; however, it turns out that an alternating series must converge if its terms consistently shrink in size and approach zero. The Alternating Series Test tells us how to show that an alternating series converges.

## Alternating Series Test

If $a_{n}>0$, then an alternating series $\sum_{n=1}^{\infty}(-1)^{n} a_{n}$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_{n}$ converges if both of the following conditions are satisfied:

1) $\lim _{n \rightarrow \infty} a_{n}=0$
2) $\left\{a_{n}\right\}$ is a decreasing sequence; that is, $a_{n+1}<a_{n}$ for all $n$.

Note: This does not say that if $\lim _{n \rightarrow \infty} a_{n} \neq 0$, the series diverges by the Alternating Series Test. The Alternating Series Test can only be used to prove convergence. If $\lim _{n \rightarrow \infty} a_{n} \neq 0$, then the series diverges by the nth Term Test for Divergence, not by the Alternating Series Test.

The Alternating Series Remainder tells us how to estimate the error found when we find the sum of the first $n$ terms of a convergent alternating series.

## Alternating Series Remainder

Suppose an alternating series satisfies the conditions of the Alternating Series Test:
namely, that $\lim _{n \rightarrow \infty} a_{n}=0$ and $\left\{a_{n}\right\}$ is a decreasing sequence $\left(a_{n+1}<a_{n}\right)$. If the series has a sum S , then
Remainder $=\left|R_{n}\right|=\left|S-S_{n}\right| \leq a_{n+1}$, where $S_{n}$ is the nth partial sum of the series.
In other words, if an alternating series satisfies the conditions of the Alternating Series Test, you can approximate the sum of the series by using the nth partial sum, $S_{n}$, and your error will have an absolute value no greater than the first term left off, $a_{n+1}$

Ex. Approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ by using its first six terms, and find the error.
Use your results to find an interval in which S must lie.

Ex. Approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4}}$ with an error less than 0.001 .

## Lagrange Form of the Remainder (also called Lagrange Error Bound or Taylor's Theorem Remainder)

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.
$f(x)=P_{n}(x)+R_{n}(x)$ so $R_{n}(x)=f(x)-P_{n}(x)$
Written in words:
Function = Polynomial + Remainder so Remainder $=$ Function - Polynomial
Taylor's Theorem: If a function $f$ is differentiable through order $n+1$ in an interval containing $c$, then for each x in the interval, there exists a number z between x and c such that
$f(x)=f(c)+f^{\prime}(c)(x-c)+\frac{f^{\prime \prime}(c)}{2!}(x-c)^{2}+\ldots+\frac{f^{(n)}(x)}{n!}(x-c)^{n}+R_{n}(x)$
where the remainder $R_{n}(x)$ (or error) is given by $R_{n}(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ (The Lagrange Remainder)
Historically the remainder was not due to Taylor but to a French mathematician, Joseph Louis Lagrange (1736-1813). For this reason, $R_{n}(x)$ is called the Lagrange form of the remainder.
When applying Taylor's Formula, we would not expect to be able to find the exact value of $z$. Rather, we would attempt to find bounds for the derivative $f^{(n+1)}(z)$ from which we will be able to tell how large the remainder $R_{n}(x)$ is. Thus, for the purpose of approximating values of a function, we restate Taylor's Formula in the following way:

## Taylor's I nequality

Suppose that $P_{n}(x)$ is the nth-degree polynomial approximation for the function $f$ about $\mathrm{x}=\mathrm{c}$ and M is the maximum value of $f^{(n+1)}(x)$ on the interval $[c, b]$ (or $[b, c]$ if $b<c$ ). Then the error in using the polynomial value $P_{n}(b)$ to estimate $f(b)$ is bounded by $\frac{M}{(n+1)!}|b-c|^{n+1}$. That is, the remainder $R_{n}(x)$ in Taylor's Formula satisfies the inequality $\left|R_{n}(x)\right| \leq\left|\frac{M}{(n+1)!}(b-c)^{n+1}\right|$

Ex. 1 Let $f$ be a function with 5 derivatives on the interval [2,3] and assume that $\left|f^{(5)}(x)\right|<0.2$ for all x in the interval [2,3]. If a fourth-degree Taylor polynomial for $f$ at $\mathrm{c}=2$ is used to estimate $f(3)$, how accurate is this approximation? Give three decimal places.

Ex. 2 (a) Find the fifth-degree Maclaurin polynomial for $\sin x$. Then use your polynomial to approximate $\sin 1$, and use Taylor's Theorem to find the maximum error for your approximation. Give three decimal places.
(b) Find an interval $[\mathrm{a}, \mathrm{b}]$ such that $a \leq \sin 1 \leq b$. Give three decimal places.
(c) Could sin 1 equal 0.9?

Ex. 3 (a) Write the fourth-degree Maclaurin polynomial for $f(x)=e^{x}$. Then use your polynomial to approximate e , and find a Lagrange error bound for the maximum error when $|x| \leq 1$. Give three decimal places,
(b) Find an interval $[\mathrm{a}, \mathrm{b}]$ such that $\mathrm{a}<\mathrm{e}<\mathrm{b}$. Give three decimal places.

Ex. 4 The function $f$ has derivatives of all orders for all real numbers x . Assume that $f(2)=6, f^{\prime}(2)=4, f^{\prime \prime}(2)=-7, f^{\prime \prime \prime}(2)=8$
(a) Write the third-degree Taylor polynomial for $f$ about $x=2$, and use it to approximate $f(2.3)$. Give three decimal places.
(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 9$ for all x in the closed interval [2, 2.3]. Use the Lagrange error bound on the approximation of $f(2.3)$ found in part (a) to find an interval [a, b] such that $a \leq f(2.3) \leq b$. Give three decimal places.
(c) Could $f(2.3)$ equal 6.922 ? Show why or why not.

## CALCULUS BC

## POWER SERIES \& ERRORS WORKSHEET

1. (a) Find the fourth-degree Taylor polynomial for $\cos x$ about $x=0$. Then use your polynomial to approximate the value of $\cos 0.8$, and use Taylor's Theorem to determine the accuracy of the approximation. Give three decimal places.
(b) Find an interval $[\mathrm{a}, \mathrm{b}]$ such that $a \leq \cos (0.8) \leq b$.
(c) Could $\cos (0.8)$ equal 0.695 ? Show why or why not.
2. (a) Write the fourth-degree Maclaurin polynomial for $f(x)=e^{x}$. Then use your polynomial to approximate $e^{-1}$, and find a Lagrange error bound for the maximum error when $|x| \leq 1 \mid$. Give three decimal places.
(b) Find an interval $[\mathrm{a}, \mathrm{b}]$ such that $a \leq e^{-1} \leq b$.
3. Let $f$ be a function that has derivatives of all orders for all real numbers $\times$ Assume that

$$
f(5)=6, f^{\prime}(5)=8, f^{\prime \prime}(5)=30, f^{\prime \prime \prime}(5)=48, \text { and }\left|f^{(4)}(x)\right| \leq 75
$$

for all x in the interval $[5,5.2]$.
(a) Find the third-degree Taylor polynomial about $\mathrm{x}=5$ for $f(x)$
(b) Use your answer to part (a) to estimate the value of $f(5.2)$. What is the maximum possible error in making this estimate? Give three decimal places.
(c) Find an interval [a, b] such that $a \leq f(5.2) \leq b$. Give three decimal places.
(d) Could $f(5.2)$ equal 8.254 ? Show why or why not.
(e) Let $g(x)=x \cdot f\left(x^{2}\right)$. Find the Maclaurin series for $g(x)$. (Write as many nonzero terms as possible.)
(f) Let $\mathrm{h}(\mathrm{x})$ be a function that has the properties $h(0)=5$ and $h^{\prime}(x)=f(x)$. Find the Maclaurin series for $h(x)$. (Write as many terms as possible.)
4. The Taylor series about $\mathrm{x}=3$ for a certain function $f$ converges to $f(x)$ for all x in the interval of convergence. The nth derivative of $f$ at $x=3$ is given by $f^{(n)}(3)=\frac{(-1)^{n} n!}{5^{n}(n+3)}$ and $f(3)=\frac{1}{3}$
(a) Write the fourth-degree Taylor polynomial for $f$ about $x=3$.
(b) Find the radius of convergence of the Taylor series for $f$ about $x=3$.
(c) Show that the third-degree Taylor polynomial approximates $f(4)$ with an error less than $\frac{1}{4000}$
5. The Taylor series about $x=5$ for a certain function $f$ converges to $f(x)$ for all $x$ in the interval of convergence. The nth derivative of $f$ at $x=5$ is given by $f^{(n)}(5)=\frac{(-1)^{n} n!}{2^{n}(n+2)}$ and $f(5)=\frac{1}{2}$. Show that the sixth-degree Taylor polynomial for $f$ about $x=5$ approximates $f(6)$ with an error less than $\frac{1}{1000}$.
6. Let f be a function that has derivatives of all orders on the interval ( $-1,1$ ). Assume $f(0)=1, f^{\prime}(0)=\frac{1}{2}, f^{\prime \prime}(0)=-\frac{1}{4}, f^{\prime \prime \prime}(0)=\frac{3}{8}$, and $\left|f^{(4)}(x)\right| \leq 6$ for all x in the interval $(0,1)$.
(a) Find the third-degree Taylor polynomial about $x=0$ for the function $f$
(b) Use your answer to part (a) to estimate the value of $f(0.5)$.
(c) What is the maximum possible error for the approximation made in part (b)?
7. Let f be the function defined by $f(x)=\sqrt{x}$.
(a) Find the second-degree Taylor polynomial about $x=4$ for the function $f$.
(b) Use your answer to part (a) to estimate the value of $f(4.2)$.
(c) Find a bound on the error for the approximation in part (b).
8. Let $f(x)=\sum_{n=0}^{\infty} \frac{x^{n}}{2^{n}}$ for all x for which the series converges.
(a) Find the interval of convergence of this series.
(b) Use the first three terms of this series to approximate $f\left(-\frac{1}{2}\right)$.
(c) Estimate the error involved in the approximation in part (b). Show your reasoning.
9. Let f be the function given by $f(x)=\cos \left(3 x+\frac{\pi}{6}\right)$ and let $\mathrm{P}(\mathrm{x})$ be the fourth-degree Taylor polynomial for $f$ about $x=0$.
(a) Find $P(x)$.
(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right)-P\left(\frac{1}{6}\right)\right|<\frac{1}{3000}$
10. Use series to find an estimate for $\int_{0}^{1} e^{-x^{2}} d x$ that is accurate to three decimal places. Justify.
11. Suppose a function f is approximated with a fourth-degree Taylor polynomial about $\mathrm{x}=1$. If the maximum value of the fifth derivative between $x=1$ and $x=3$ is 0.01 , that is, $\left|f^{(5)}(x)\right| \leq 0.01$, then the maximum error incurred using this approximation to compute $f(3)$ is
(A) 0.054
(B) 0.0054
(C) 0.26667
(D) 0.02667
(E) 0.00267
12. The maximum error incurred by approximating the sum of the series $1-\frac{1}{2!}+\frac{2}{3!}-\frac{3}{4!}+\frac{4}{5!}-\ldots$ by the sum of the first six terms is
(A) 0.001190
(B) 0.006944
(C) 0.33333
(D) 0.125000
(E) None of these

1. (a) $1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!} ; 0.003$
(b) $0.694 \leq \cos (0.8) \leq 0.700$
(c) Yes, $\cos (0.8)$ can equal 0.695 since it lies in the interval found in (b).
2. (a) $1+x+\frac{x^{2}}{2}+\frac{x^{3}}{3}+\frac{x^{4}}{4} ; 0.375 ; 0.025$
(b) $0.352 \leq e^{-1} \leq 0.398$
3. (a) $6+9(x-5)+15(x-5)^{2}+8(x-5)^{3}$
(b) 8.264
(c) $8.259 \leq f(5.2) \leq 8.269$
(d) No, f(5.2) can't equal 8.254 because it isn't in the interval found in (c).
(e)
4. (a) $\frac{1}{3}-\frac{(x-3)}{20}+\frac{(x-3)^{2}}{125}-\frac{(x-3)^{3}}{750}+\frac{(x-3)^{4}}{4375}$
(b) 5
(c) $\mid$ Error $\left\lvert\,<\frac{1}{4375}<\frac{1}{4000}\right.$

Since this is a convergent alternating series with decreasing terms, we can use the Alternating Series Remainder, so that the error is less than the first truncated term.
5. $f(6)=\frac{1}{2}-\frac{1}{6}+\frac{1}{16}-\frac{1}{40}+\frac{1}{96}-\frac{1}{224}+\frac{1}{512}-\frac{1}{1152}+\ldots$

This is an alternating series whose terms are decreasing in size so the error involved in approximating $f(6)$ with the sixth-degree Taylor polynomial is less in magnitude than the seventh-degree term.
$\mid$ Error $\left\lvert\,<\frac{1}{1152}<\frac{1}{1000}\right.$ by the Alternating Series
Remainder
6. (a) $1+\frac{1}{2} x-\frac{1}{8} x^{2}+\frac{1}{16} x^{3}$
(b) $\frac{157}{128}=1.22656$
(c) The error is at most $\frac{1}{64}=0.015625$
7. (a) $2+\frac{x-4}{4}-\frac{(x-4)^{2}}{64}$
(b) 2.049375
(c) The maximum value of the third derivative $f^{\prime \prime}(x)=\frac{3}{8 x^{5 / 2}}$ on $[4,4.2]$ occurs at $x=4$ and is $\frac{3}{256}$. Then $\left|R_{2}(x)\right| \leq \frac{\frac{3}{256}}{3!}(0.2)^{3}=1.5625 \times 10^{-5}$
8. (a) $(-2,2)$
(b) $1-\frac{1}{4}+\frac{1}{16}=\frac{13}{16}$
(c) By the Alternating Series Remainder, the error is at most the first omitted term, $\frac{1}{64}$
9. (a) $P(x)=\frac{\sqrt{3}}{2}-\frac{3 x}{2}-\frac{9 \sqrt{3} x^{2}}{2 \cdot 2!}+\frac{27 x^{3}}{2 \cdot 3!}+\frac{81 \sqrt{3} x^{4}}{2 \cdot 4!}$
(b) $\left|R_{4}(x)\right|=\left|\frac{f^{(5)}(z)(x-0)^{5}}{5!}\right| \leq\left|\frac{243 x^{5}}{5!}\right|$ so
$\left|R_{4}\left(\frac{1}{6}\right)\right| \leq\left(\frac{243}{5!}\right)\left(\frac{1}{6}\right)^{5}=\frac{1}{5!2^{5}}=\frac{1}{(120)(32)}<\frac{1}{3000}$
10. Because this is an alternating series whose terms decrease in value, we can truncate after 6 terms and have an error correct to three decimal places.
$\int_{0}^{1} e^{-x^{2}} d x \approx 1-\frac{1}{3}+\frac{1}{5(2!)}-\frac{1}{7(3!)}+\frac{1}{9(4!)}-\frac{1}{11(5!)}=$
0.746729
11. E
12. A

