CH 8 FR ANSWERS

1.

a.

$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$$

b.

No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.

c.

$$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$$

d.

$$\ln\left(\frac{5}{4}\right) = \ln\left(1+\frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$$

Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$.

Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0,

$$\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3}\cdot\frac{1}{64} < \frac{1}{100}.$$

2.

a.

$$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{6(-1)^n x^n}{n!3^n} + \cdots$$

b.

$$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{4(27)} + \cdots + \frac{6(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots$$

c.

$$a = -3 \text{ and } k = -\frac{1}{2}$$

3.

a.

$$f'(x) = -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{3n-1} + \cdots$$

b.

$$-\frac{16}{27}.$$

c.

$$x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + \frac{(-1)^n x^{3n+1}}{3n+1} + \cdots$$

d.

$$\int_0^{1/2} \frac{1}{1+t^3}\,dt \approx \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}.$$

The series in part (c) with $x = \frac{1}{2}$ has terms that alternate, decrease in absolute value, and have limit 0. Hence the error is bounded by the absolute value of the next term.

$$\left|\int_0^{1/2} \frac{1}{1+t^3}\,dt - \left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}\right)\right| < \frac{\left(\frac{1}{2}\right)^{10}}{10} = \frac{1}{10240} < 0.0001$$

4.

a.

f has a relative maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

b.

$$P(x) = 6 - \frac{3!x^2}{5^2\,2!} + \frac{4!x^3}{5^3\,2^2\,3!} = 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3$$

c.

The radius of convergence is 5.

5.

a.

$$f(2) = T(2) = 7$$

$$\frac{f''(2)}{2!} = -9 \text{ so } f''(2) = -18$$

b.

Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$.
Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.

c.

$f(0) \approx T(0) = -5$

It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.

d.

Lagrange error bound $= \frac{6}{4!}|0-2|^4 = 4$

$f(0) \le T(0) + 4 = -1$

Therefore, $f(0)$ is negative.

6.

a.

$$= 1 + \frac{2}{3}(x-2) + \frac{3}{3^2}(x-2)^2 + \frac{4}{3^3}(x-2)^3 + \\ + \cdots + \frac{n+1}{3^n}(x-2)^n + \cdots$$

b.

The radius of convergence is 3.

c.

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \\ + \cdots + \frac{1}{3^n}(x-2)^{n+1} + \cdots$$

d.

No, the Taylor series does not converge at $x = -2$ because the geometric series only converges on the interval $|x-2| < 3$.

7.

a.

$$-\frac{1}{3} < x \le \frac{1}{3}.$$

b.

$$\ln\left(\frac{1}{2}\right)$$

c.

Some p such that $0 < p \le \frac{1}{2}$ because $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by AST, but the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges for $2p \le 1$.

d.

Some p such that $\frac{1}{2} < p \le 1$ because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.

8.

a.

$$\frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$$

b.

The radius of convergence is 2.

c.

The Taylor series about $x = 5$ for the function f, when evaluated at $x = 6$, is an alternating series with absolute value of terms decreasing to 0. The error in approximating $f(6)$ with the 6th degree Taylor polynomial at $x = 6$ is less than the first omitted term in the series.

$$|f(6) - P_6(f,5)(6)| \le \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$$

9.

a.

$$f'(0) = a_1 = \frac{1}{2}$$

$$f^{(17)}(0) = 17!a_{17} = 17!\left(\frac{1}{18!}\right) = \frac{1}{18}$$

b.

Converges for all x , by ratio test

c.

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$$

d.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^x - 1 = g(x) = xf(x)$$

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$$

10.

a.

$$T_2(x) = 3 + (-2)(x-1) + \frac{2}{2}(x-1)^2$$

$$f(0.7) \approx 3 + 0.6 + 0.09 = 3.69$$

b.

$$T_3(x) = 3 - 2(x-1) + (x-1)^2 + \frac{4}{6}(x-1)^3$$

$$f(1.2) \approx 3 - 0.4 + 0.04 + \frac{2}{3}(0.008) = 2.645$$

c.

$$T_3'(x) = -2 + 2(x-1) + 2(x-1)^2$$

$$f'(1.2) \approx -2 + 0.4 + 0.08 = -1.52$$

11.

a.

$$e^{-2x^2} = 1 - 2x^2 + \frac{4x^4}{2} - \frac{8x^6}{3!} + \cdots + \frac{(-1)^n 2^n x^{2n}}{n!} + \cdots$$

b.

$$-\infty < x < \infty$$

12.

a.

$$= 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 3!} + \cdots + \frac{x^n}{2^n n!} + \cdots$$

b.

$$= \frac{1}{2} + \frac{x}{2^2 2!} + \frac{x^2}{2^3 3!} + \cdots + \frac{x^{n-1}}{2^n n!} + \cdots$$

c.

$$g'(x) = \frac{1}{2^2 2!} + \frac{2x}{2^3 3!} + \cdots + \frac{(n-1)x^{n-2}}{2^n n!} + \cdots$$

$$= \frac{1}{8} + \frac{x}{24} + \cdots + \frac{(n-1)x^{n-2}}{2^n n!} + \cdots$$

$$g'(2) = \frac{1}{2^2 2!} + \frac{2 \cdot 2}{2^3 3!} + \cdots + \frac{(n-1)2^{n-2}}{2^n n!} + \cdots$$

$$= \frac{1}{8} + \frac{1}{12} + \cdots + \frac{n-1}{4n!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{n}{4(n+1)!}$$

Also, $g(x) = \dfrac{e^{x/2} - 1}{x}$

$$g'(x) = \frac{x \cdot \frac{1}{2} e^{x/2} - (1)\left(e^{x/2} - 1\right)}{x^2}$$

$$g'(2) = \frac{2 \cdot \frac{1}{2} e - (e-1)}{4} = \frac{1}{4}$$

13.

a.

$0 < \dfrac{1}{n^p \ln(n)} < \dfrac{1}{n^p}$ for $\ln(n) > 1$, for $n \geq 3$

by p-series test, $\sum \dfrac{1}{n^p}$ converges if $p > 1$

and by direct comparison, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$ converges.

b.

Let $f(x) = \dfrac{1}{x \ln x}$, so series is $\displaystyle\sum_{n=2}^{\infty} f(n)$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b \to \infty} \ln\left|\ln x\right|\Big|_2^b = \lim_{b \to \infty} [\ln(\ln(b)) - \ln(\ln 2)] = \infty$$

Since $f(x)$ monotonically decreases to 0, the integral test shows $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

c

$\dfrac{1}{n^p \ln n} > \dfrac{1}{n \ln n} > 0$ for $p < 1$,

so by direct comparison, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges for $0 \leq p < 1$

14.

a.

$$f(t) = 4 - 4t^2 + 4t^4 - 4t^6 + \cdots + (-1)^n 4t^{2n} + \cdots$$

b.

$$= 4x - \frac{4}{3}x^3 + \frac{4}{5}x^5 - \frac{4}{7}x^7 + \cdots + \frac{(-1)^n 4x^{2n+1}}{2n+1} + \cdots$$

c.

Converges for $-1 \le x \le 1$.

15.

a.

$-2 < x < 0.$

b.

$-\frac{1}{x}$

c.

$\ln 2$

d.

$h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$

16.

a.

$$\frac{1}{x-1} = 1-(x-2)+(x-2)^2-(x-2)^3+\cdots+(-1)^n(x-2)^n+\cdots$$

b.

$$\ln|x-1| = C + x - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \cdots + \frac{(-1)^n(x-2)^{n+1}}{n+1} + \cdots$$

$$0 = \ln|2-1| \Rightarrow C = -2$$

17.

a.

$$P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$$

b.

$$\frac{-5^{22}\sqrt{2}}{2(22!)}$$

c.

$$\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| \le \max_{0 \le c \le \frac{1}{10}} \left|f^{(4)}(c)\right| \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4$$

$$\le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$$

d.

$$= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$$

18.

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

(b) $P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max_{1.9 \le x \le 2} \left|h^{(4)}(x)\right| = \frac{584}{9}$.

Therefore, $|h(1.9) - P_3(1.9)| \le \frac{584}{9}\frac{|1.9-2|^4}{4!}$

$= 2.7037 \times 10^{-4}$

$< 3 \times 10^{-4}$

19.

a)

$$x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$$

b)

$-1 < x \le 1.$

c)

$$f'\left(t^2\right) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$$

$g(1) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}.$

d)

$$\left| g(1) - \frac{18}{55} \right| < \frac{3 \cdot 1^{17}}{17} = \frac{3}{17} < \frac{1}{5}.$$

20.

a)

$$\frac{1}{2}.$$

b)

$$\left(-\frac{1}{2}, \frac{1}{2}\right].$$

c)

$$y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$$

$$= 4x^2 - 4x^3 + \frac{16}{3}x^4 - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$$

$$y' = 8x - 12x^2 + \frac{64}{3}x^3 - \cdots + \frac{(-1)^n n (2x)^{n-1} \cdot 2}{n-1} + \cdots$$

$$xy' = 8x^2 - 12x^3 + \frac{64}{3}x^4 - \cdots + \frac{(-1)^n n (2x)^n}{n-1} + \cdots$$

$$xy' - y = 4x^2 - 8x^3 + 16x^4 - \cdots + (-1)^n (2x)^n + \cdots$$

$$= 4x^2 \left(1 - 2x + 4x^2 - \cdots + (-1)^n (2x)^{n-2} + \cdots\right)$$