

Chapter 8 AP Problems

1. NO CALCULATOR

Let f be the function given by $f(x) = \frac{2x}{1+x^2}$.

- (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.
- (b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.
- (c) The derivative of $\ln(1+x^2)$ is $\frac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln(1+x^2)$ about $x = 0$.
- (d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

2. NO CALCULATOR

Let f be the function given by $f(x) = 6e^{-x/3}$ for all x .

- (a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.
- (b) Let g be the function given by $g(x) = \int_0^x f(t) dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.
- (c) The function h satisfies $h(x) = kf'(ax)$ for all x , where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k .

3. NO CALCULATOR

The function f is defined by $f(x) = \frac{1}{1+x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

- (a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.
- (b) Use your results from part (a) to find the sum of the infinite series $-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots + (-1)^n \frac{3n}{2^{3n-1}} + \cdots$.
- (c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t) dt$.
- (d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t) dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t) dt$ that guarantee that this approximation is within $\frac{1}{10,000}$ of the exact value of the integral?

4. NO CALCULATOR

The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The n th derivative of f at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1} (n+1)!}{5^n (n-1)^2} \text{ for } n \geq 2.$$

The graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

- (a) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.
- (b) Write the third-degree Taylor polynomial for f about $x = 0$.
- (c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

5. NO CALCULATOR

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

- (a) Find $f(2)$ and $f''(2)$.
- (b) Is there enough information given to determine whether f has a critical point at $x = 2$?
If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.
- (c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.
- (d) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \leq 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

6. NO CALCULATOR

The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The n th derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n+1)!}{3^n}$ for $n \geq 1$, and $f(2) = 1$.

- (a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.
- (b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.
- (c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x . Write the first four terms and the general term of the Taylor series for g about $x = 2$.
- (d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

7. NO CALCULATOR

The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \leq x < 1$.

- Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.
- Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.
- Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.
- Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

8. CALCULATOR

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The n th derivative of f at $x = 5$ is given by $f^{(n)}(5) = \frac{(-1)^n n!}{2^n(n+2)}$, and $f(5) = \frac{1}{2}$.

- Write the third-degree Taylor polynomial for f about $x = 5$.
- Find the radius of convergence of the Taylor series for f about $x = 5$.
- Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\frac{1}{1000}$.

9. CALCULATOR

The Maclaurin series for $f(x)$ is given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$

- Find $f'(0)$ and $f^{(17)}(0)$.
- For what values of x does the given series converge? Show your reasoning.
- Let $g(x) = xf(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.
- Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

10. NO CALCULATOR

Let f be a function that has derivatives of all orders for all real numbers.

Assume $f(1) = 3$, $f'(1) = -2$, $f''(1) = 2$, and $f'''(1) = 4$.

- Write the second-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(0.7)$.
- Write the third-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(1.2)$.
- Write the second-degree Taylor polynomial for f' , the derivative of f , about $x = 1$ and use it to approximate $f'(1.2)$.

11. NO CALCULATOR

Let f be the function given by $f(x) = e^{-2x^2}$.

- Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x = 0$.
- Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.
- Let g be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x = 0$. Show that $|f(x) - g(x)| < 0.02$ for $-0.6 \leq x \leq 0.6$.

12. NO CALCULATOR

Let f be the function given by $f(x) = e^{\frac{x}{2}}$.

- Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$ about $x = 0$.
- Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about $x = 0$ for $g(x) = \frac{e^{\frac{x}{2}} - 1}{x}$.
- For the function g in part (b), find $g'(2)$ and use it to show that $\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$.

13. NO CALCULATOR

Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$, where $p \geq 0$.

- Show that the series converges for $p > 1$.
- Determine whether the series converges or diverges for $p = 1$. Show your analysis.
- Show that the series diverges for $0 \leq p < 1$.

14. NO CALCULATOR

Let f be the function given by $f(t) = \frac{4}{1+t^2}$ and G be the function given by

$$G(x) = \int_0^x f(t) dt.$$

- Find the first four nonzero terms and the general term for the power series expansion of $f(t)$ about $t = 0$.
- Find the first four nonzero terms and the general term for the power series expansion of $G(x)$ about $x = 0$.
- Find the interval of convergence of the power series in part (b). (Your solution must include an analysis that justifies your answer.)

15. NO CALCULATOR

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

- Find the interval of convergence of the power series for f . Justify your answer.
- The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f .
- Let g be the function defined by $g(x) = \int_{-1}^x f(t) dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.
- Let h be the function defined by $h(x) = f(x^2 - 1)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

16. NO CALCULATOR

Let f be the function defined by $f(x) = \frac{1}{x-1}$.

- (a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.
- (b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

17. NO CALCULATOR

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

- (a) Find $P(x)$.
- (b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.
- (c) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.
- (d) Let G be the function given by $G(x) = \int_0^x f(t) dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

18. CALCULATOR

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \leq x \leq 3$.

- (a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.
- (b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.
- (c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4} .

19. NO CALCULATOR

Let $f(x) = \ln(1 + x^3)$.

- The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f .
- The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.
- Write the first four nonzero terms of the Maclaurin series for $f'(t^2)$. If $g(x) = \int_0^x f'(t^2) dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.
- The Maclaurin series for g , evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

20. NO CALCULATOR

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

- Find the interval of convergence for the Maclaurin series of f . Justify your answer.
- Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).