AP Calculus Chapter 5 Free Response Review

#1

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.
(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.
(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)dt$. Find the value $w'(3)$.
(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

#2

A particle moves on the x-axis so that its acceleration at any time $t > 0$ is given by $a = \frac{t}{8} - \frac{1}{t^2}$. When $t = 1$, $v = \frac{9}{16}$, and $s = \frac{25}{48}$.

(a) Find the velocity v in terms of t.

(b) Does the numerical value of the velocity ever exceed 50? Explain.

(c) Find the distance s from the origin at time $t = 2$.

#3

A particle moves along the x-axis in such a way that at time $t > 0$ its position coordinate is $x = \sin(e^t)$.

(a) Find the velocity and acceleration of the particle at time t.

(b) At what time does the particle first have zero velocity?

(c) What is the acceleration of the particle at the time determined in part (b)?

#4

Let $y = 2e^{\cos x}$.

(a) Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) If x and y both vary with time in such a way that y increases at a steady rate of 5 units per second, at what rate is x changing when $x = \frac{\pi}{2}$?

#5

Let f be the function given by $f(x) = \ln \frac{x}{x-1}$.

(a) What is the domain of f ?

(b) Find the value of the derivative of f at $x = -1$.

(c) Write an expression for $f^{-1}(x)$, where f^{-1} denotes the inverse function of f.

#6

Let f be the function defined by $f(x) = (x^2 + 1)\, e^{-x}$ for $-4 \leq x \leq 4$.

(a) For what value of x does f reach its absolute maximum? Justify your answer.

(b) Find the x-coordinates of all points of inflection of f. Justify your answer.

#7

A particle moves along the x-axis with acceleration given by $a(t) = 2t - 10 + \frac{12}{t}$ for $t \geq 1$.

(a) Write an expression for the velocity $v(t)$, given that $v(1) = 9$.

(b) For what values of t, $1 \leq t \leq 3$, is the velocity a maximum? Justify your answer.

Answers

#1

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$

$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$

Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$

Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

(d) $g(1) = 2$, so $g^{-1}(2) = 1$.

$$\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$$

An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.

#2

(a) $v = \int \left(\frac{t}{8} - \frac{1}{t^2} \right) dt = \frac{t^2}{16} + \frac{1}{t} + C$

At $t = 1$, $\frac{9}{16} = \frac{1}{16} + 1 + C$ and so $C = -\frac{1}{2}$. Therefore $v = \frac{t^2}{16} + \frac{1}{t} - \frac{1}{2}$.

(b) Yes, the numerical value does exceed 50. This can be explained by any one of the following observations:

(i) $\lim_{t \to +\infty} v(t) = \infty$

(ii) $\lim_{t \to 0^+} v(t) = \infty$

(iii) When $t = 32$, for example, $v = \frac{32^2}{16} + \frac{1}{32} - \frac{1}{2} = 64 + \frac{1}{32} - \frac{1}{2} > 50$

(c) $s = \int \left(\frac{t^2}{16} + \frac{1}{t} - \frac{1}{2} \right) dt = \frac{t^3}{48} + \ln t - \frac{1}{2} t + C$

At $t = 1$, $\frac{25}{48} = \frac{1}{48} - \frac{1}{2} + C$ and so $C = 1$. Therefore $s = \frac{t^3}{48} + \ln t - \frac{1}{2} t + 1$.

#3

(a) $x = \sin(e^t)$

$$v = \frac{dx}{dt} = e^t \cos(e^t)$$

$$a = \frac{dv}{dt} = e^t(\cos(e^t) - e^t \sin(e^t))$$

(b) $v(t) = 0$ when $\cos(e^t) = 0$. Hence $e^t = \frac{\pi}{2}$ gives the first time when the velocity is zero, and so $t = \ln\frac{\pi}{2}$.

(c) $a\left(\ln\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2}\right) = -\frac{\pi^2}{4}$

#4

(a) $y = 2e^{\cos x}$

$$\frac{dy}{dx} = 2e^{\cos x}(-\sin x) = -2(\sin x)e^{\cos x}$$

$$\frac{d^2y}{dx^2} = -2(\sin x)e^{\cos x}(-\sin x) - 2(\cos x)e^{\cos x} = 2e^{\cos x}(\sin^2 x - \cos x)$$

(b) $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = -2(\sin x)e^{\cos x} \cdot \frac{dx}{dt}$

Substituting $\frac{dy}{dt} = 5$ and $x = \frac{\pi}{2}$ gives

$$5 = -2\left(\sin\frac{\pi}{2}\right)e^{\cos(\pi/2)}\frac{dx}{dt} = -2(1)e^0\frac{dx}{dt} = -2\frac{dx}{dt}.$$

Therefore $\frac{dx}{dt} = -\frac{5}{2}$ when $x = \frac{\pi}{2}$.

#5

(a) $\dfrac{x}{x-1} > 0$

$x > 0$ and $x - 1 > 0 \Rightarrow x > 1$

$x < 0$ and $x - 1 < 0 \Rightarrow x < 0$

$x < 0$ or $x > 1$

(b) $f'(x) = \dfrac{x-1}{x} \cdot \dfrac{(x-1) - x}{(x-1)^2}$

$= \dfrac{-1}{x(x-1)}$

or

$\ln|x| - \ln|x-1| \Rightarrow f'(x) = \dfrac{1}{x} - \dfrac{1}{x-1}$

$f'(-1) = -\dfrac{1}{2}$

(c) $y = \ln\left(\dfrac{x}{x-1}\right)$

$e^y = \dfrac{x}{x-1}$

$x(e^y - 1) = e^y$

$x = \dfrac{e^y}{e^y - 1}$

$f^{-1}(x) = \dfrac{e^x}{e^x - 1}$

#6

The absolute maximum is at a critical point or an endpoint. There is a critical point at $x = 1$.

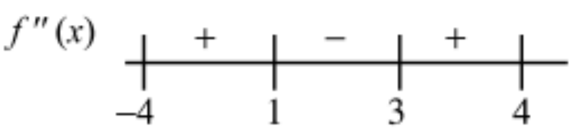
$f(-4) = 17e^4$

$f(1) = \frac{2}{e}$

$f(4) = \frac{17}{e^4}$

Therefore the absolute maximum is at $x = -4$.

(b) $f''(x) = e^{-x}(x-1)^2 - e^{-x} \cdot 2(x-1) = e^{-x}(x-1)(x-3)$



$f''(x) > 0 \qquad -4 < x < 1$

$f''(x) < 0 \qquad 1 < x < 3$

$f''(x) > 0 \qquad 3 < x < 4$

The points of inflection are at $x = 1$ and $x = 3$.

#7

(a) $v(t) = t^2 - 10t + 12\ln t + C$

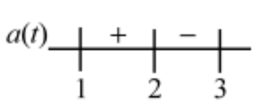
$9 = v(1) = 1 - 10 + 12(0) + C$

$C = 18$

$v(t) = t^2 - 10t + 12\ln t + 18$

(b) $a(t) = \frac{2t^2 - 10t + 12}{t} = \frac{2(t-2)(t-3)}{t}$

$a(t) = 0$ when $t = 2$ and $t = 3$.



Since the velocity is increasing for $1 \le t < 2$ and decreasing for $2 < t < 3$, the velocity is a maximum at $t = 2$.