Chapter 4 Free Response Answers

1.

(a) $v(t) = \int 4\cos 2t\, dt$

$v(t) = 2\sin 2t + C$

$v(0) = 1 \Rightarrow C = 1$

$v(t) = 2\sin 2t + 1$

(b) $x(t) = \int 2\sin 2t + 1\, dt$

$x(t) = -\cos 2t + t + C$

$x(0) = 0 \Rightarrow C = 1$

$x(t) = -\cos 2t + t + 1$

(c) $2\sin 2t + 1 = 0$

$\sin 2t = -\frac{1}{2}$

$2t = \frac{7\pi}{6}, \frac{11\pi}{6}$

$t = \frac{7\pi}{12}, \frac{11\pi}{12}$

2.

(a) $x(t) = 4t^3 - 18t^2 + 15t + C$

$0 = x(1) = 4 - 18 + 15 + C$

Therefore $C = -1$

$x(t) = 4t^3 - 18t^2 + 15t - 1$

(b) $0 = v(t) = 12t^2 - 36t + 15$

$3(2t-1)(2t-5) = 0$

$t = \frac{1}{2}, \frac{5}{2}$

(c) $\frac{dv}{dt} = 24t - 36$

$\frac{dv}{dt} = 0$ when $t = \frac{3}{2}$

$v(0) = 15$

$v\left(\frac{3}{2}\right) = -12$

$v(2) = -9$

Maximum velocity is 15

(d) Total distance $= \int_0^{1/2} v(t)\,dt - \int_{1/2}^{2} v(t)\,dt$

$= \left(x\left(\frac{1}{2}\right) - x(0)\right) - \left(x(2) - x\left(\frac{1}{2}\right)\right)$

$= \frac{5}{2} - (-1) - \left(-11 - \frac{5}{2}\right) = 17$

3.

(a) $x(t) = \int v(t)\,dt = \int (3t^2 - 2t - 1)\,dt$

$= t^3 - t^2 - t + C$

$x(2) = 8 - 4 - 2 + C = 5;\quad C = 3$

$x(t) = t^3 - t^2 - t + 3$

(b) avg. vel. $= \frac{x(3) - x(0)}{3 - 0}$

$= \frac{18 - 3}{3} = 5$

$3t^2 - 2t - 1 = 5$

$t = \frac{1 + \sqrt{19}}{3}$ or 1.786

c)

$v(t) = 3t^2 - 2t - 1 = 0$

$t = -\frac{1}{3},\ t = 1$

$x(0) = 3$

$x(1) = 1 - 1 - 1 + 3 = 2$

$x(3) = 27 - 9 - 3 + 3 = 18$

distance $= (3 - 2) + (18 - 2) = 17$

4.

(a) $v(t) = \sin(t) + C$

$2 = \sin(0) + C$

$C = 2$

$v(t) = \sin(t) + 2$

(b) $x(t) = -\cos(t) + 2t + C$

$5 = -\cos(0) + 2(0) + C$

$C = 6$

$x(t) = -\cos(t) + 2t + 6$

(c) The particle moves to the right when $v(t) > 0$,. i.e. when $\sin(t) + 2 > 0$. This is true for all $t \ge 0$ because

$-1 \le \sin(t) \le 1 \Rightarrow 0 < -1 + 2 \le \sin(t) + 2 \le 1 + 2$ for all t.

(d) The particle never changes directions since it moves to the right for all $t \ge 0$.

$x(0) = -\cos(0) + 2(0) + 6 = 5$

$x\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) + 6 = \pi + 6$

Distance $= x\left(\frac{\pi}{2}\right) - x(0) = \pi + 1$

or

Distance $= \int_0^{\pi/2} |v(t)|\,dt = \int_0^{\pi/2} |\sin(t) + 2|\,dt$

$= \int_0^{\pi/2} (\sin(t) + 2)\,dt = (-\cos t + 2t)\Big|_0^{\pi/2} = \pi + 1$

5.

(a) $v(t)=3t^2-12t+9$

$a(t)=6t-12$

a is increasing, so a is minimum at $t=0$

$a(0)=-12$ is minimum value of a.

(b) Method 1:

$v(t)$: $+$ on $(0,1)$, $-$ on $(1,3)$, $+$ on $(3,5)$ (number line marked 0, 1, 3, 5)

$$d=\int_0^1(3t^2-12t+9)\,dt-\int_1^3(3t^2-12t+9)\,dt+\int_3^5(3t^2-12t+9)\,dt$$

$$=\left[t^3-6t^2+9t\right]_0^1-\left[t^3-6t^2+9t\right]_1^3+\left[t^3-6t^2+9t\right]_3^5$$

$$=4-(-4)+20=28$$

or

Method 2: $x(t)=t^3-6t^2+9t-2$

$\left[\text{or } x(t)=t^3-6t^2+9t+C\right]$

$x(0)=-2$

$x(1)=2$

$x(3)=-2$

$x(5)=18$

Total distance $=4+4+20=28$

(c) Method 1:

$$\frac{\int_0^5(3t^2-12t+9)\,dt}{5-0}$$

$$=\frac{1}{5}\left[t^3-6t^2+9t\right]_0^5=\frac{1}{5}(20)=4$$

or

Method 2: $\dfrac{x(5)-x(0)}{5-0}=\dfrac{18-(-2)}{5}=4$

6.

(a) $v(t)=4t^3-4t$

$v(t)=4t^3-4t=0$

$=4t(t^2-1)=0$

Therefore $t=0$, $t=1$

(b) $x(t)=t^4-2t^2+C$

$3=x(1)=1^4-2\cdot 1+C$

$3=C-1$

$4=C$

$x(t)=t^4-2t^2+4$

(c) $x(0)=4$

$x(1)=3$

$x(2)=12$

Distance $=1+9=10$

7.

(a) $a(3) = v'(3) = 6\cos 9 = -5.466$ or -5.467

(b) Distance $= \int_0^3 |v(t)|\, dt = 1.702$

OR

For $0 < t < 3$, $v(t) = 0$ when $t = \sqrt{\pi} = 1.77245$ and $t = \sqrt{2\pi} = 2.50663$

$x(0) = 5$

$x(\sqrt{\pi}) = 5 + \int_0^{\sqrt{\pi}} v(t)\, dt = 5.89483$

$x(\sqrt{2\pi}) = 5 + \int_0^{\sqrt{2\pi}} v(t)\, dt = 5.43041$

$x(3) = 5 + \int_0^3 v(t)\, dt = 5.77356$

$|x(\sqrt{\pi}) - x(0)| + |x(\sqrt{2\pi}) - x(\sqrt{\pi})| + |x(3) - x(\sqrt{2\pi})| = 1.702$

(c) $x(3) = 5 + \int_0^3 v(t)\, dt = 5.773$ or 5.774

(d) The particle's rightmost position occurs at time $t = \sqrt{\pi} = 1.772$.

The particle changes from moving right to moving left at those times t for which $v(t) = 0$ with $v(t)$ changing from positive to negative, namely at $t = \sqrt{\pi}, \sqrt{3\pi}, \sqrt{5\pi}$ $(t = 1.772, 3.070, 3.963)$.

Using $x(T) = 5 + \int_0^T v(t)\, dt$, the particle's positions at the times it changes from rightward to leftward movement are:

T:	0	$\sqrt{\pi}$	$\sqrt{3\pi}$	$\sqrt{5\pi}$
$x(T)$:	5	5.895	5.788	5.752

The particle is farthest to the right when $T = \sqrt{\pi}$.

8.

(a) g' changes from increasing to decreasing at $x = 1$;
g' changes from decreasing to increasing at $x = 4$.

Points of inflection for the graph of $y = g(x)$ occur at $x = 1$ and $x = 4$.

(b) The only sign change of g' from positive to negative in the interval is at $x = 2$.

$$g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$$
$$g(2) = 5$$
$$g(7) = 5 + \int_2^{7} g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$$

The maximum value of g for $-3 \le x \le 7$ is $\frac{15}{2}$.

(c) $\dfrac{g(7) - g(-3)}{7 - (-3)} = \dfrac{\frac{3}{2} - \frac{15}{2}}{10} = -\dfrac{3}{5}$

(d) $\dfrac{g'(7) - g'(-3)}{7 - (-3)} = \dfrac{1 - (-4)}{10} = \dfrac{1}{2}$

No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.

9.

(a) $f'(x) = 0$ at $x = -3,\ 1,\ 4$
f' changes from positive to negative at -3 and 4.
Thus, f has a relative maximum at $x = -3$ and at $x = 4$.

(b) f' changes from increasing to decreasing, or vice versa, at $x = -4,\ -1,$ and 2. Thus, the graph of f has points of inflection when $x = -4,\ -1,$ and 2.

(c) The graph of f is concave up with positive slope where f' is increasing and positive: $-5 < x < -4$ and $1 < x < 2$.

(d) Candidates for the absolute minimum are where f' changes from negative to positive (at $x = 1$) and at the endpoints ($x = -5, 5$).

$$f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \frac{\pi}{2} + 2\pi > 3$$
$$f(1) = 3$$
$$f(5) = 3 + \int_1^{5} f'(x)\,dx = 3 + \frac{3 \cdot 2}{2} - \frac{1}{2} > 3$$

The absolute minimum value of f on $[-5, 5]$ is $f(1) = 3$.

10.

(a) $g(-1) = \int_{-4}^{-1} f(t)\,dt = -\frac{1}{2}(3)(5) = -\frac{15}{2}$
$g'(-1) = f(-1) = -2$
$g''(-1)$ does not exist because f is not differentiable at $x = -1$.

(b) $x = 1$
$g' = f$ changes from increasing to decreasing at $x = 1$.

(c) $x = -1,\ 1,\ 3$

(d) h is decreasing on $[0, 2]$
$h' = -f < 0$ when $f > 0$

11.

(a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(4 + 2) = 3$
$g'(3) = f(3) = 2$
$g''(3) = f'(3) = \dfrac{0 - 4}{4 - 2} = -2$

(b)
$$\frac{g(3) - g(0)}{3} = \frac{1}{3}\int_0^3 f(t)\,dt = \frac{1}{3}\left(\frac{1}{2}(2)(4) + \frac{1}{2}(4 + 2)\right) = \frac{7}{3}$$

(c) There are two values of c.
We need $\frac{7}{3} = g'(c) = f(c)$
The graph of f intersects the line $y = \frac{7}{3}$ at two places between 0 and 3.

(d) $x = 2$ and $x = 5$
because $g' = f$ changes from increasing to decreasing at $x = 2$, and from decreasing to increasing at $x = 5$.

12.

(a) $f'(x) = 3\sqrt{4 + (3x)^2}$

$g'(x) = f'(\sin x) \cdot \cos x$

$= 3\sqrt{4 + (3\sin x)^2} \cdot \cos x$

(b) $g(\pi) = 0,\ g'(\pi) = -6$

Tangent line: $y = -6(x - \pi)$

(c) For $0 < x < \pi$, $g'(x) = 0$ only at $x = \frac{\pi}{2}$.

$g(0) = g(\pi) = 0$

$g\left(\frac{\pi}{2}\right) = \int_0^3 \sqrt{4 + t^2}\, dt > 0$

The maximum value of g on $[0, \pi]$ is

$\int_0^3 \sqrt{4 + t^2}\, dt$.

13.

(a) $\frac{(0+7)}{2} \cdot 8 + \frac{(7+8)}{2} \cdot 6 + \frac{(8+2)}{2} \cdot 8 + \frac{(2+0)}{2} \cdot 2$

$= 115 \text{ ft}^2$

(b) $\frac{1}{120}\int_0^{120} 115v(t)\, dt$

$= 1807.169$ or $1807.170 \text{ ft}^3/\text{min}$

(c) $\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right) dx = 122.230$ or 122.231 ft^2

(d) Let C be the cross-sectional area approximation from part (c). The average volumetric flow is

$\frac{1}{20}\int_{40}^{60} C \cdot v(t)\, dt = 2181.912$ or $2181.913 \text{ ft}^3/\text{min}$.

Yes, water must be diverted since the average volume for this 20-minute period exceeds $2100 \text{ ft}^3/\text{min}$.

14.

(a) Midpoint Riemann sum is

$10 \cdot [v(5) + v(15) + v(25) + v(35)]$

$= 10 \cdot [9.2 + 7.0 + 2.4 + 4.3] = 229$

The integral gives the total distance in miles that the plane flies during the 40 minutes.

(b) By the Mean Value Theorem, $v'(t) = 0$ somewhere in the interval $(0, 15)$ and somewhere in the interval $(25, 30)$. Therefore the acceleration will equal 0 for at least two values of t.

(c) $f'(23) = -0.407$ or -0.408 miles per minute^2

(d) Average velocity $= \frac{1}{40}\int_0^{40} f(t)\, dt$

$= 5.916$ miles per minute

15.

(a) $\int_{30}^{60} |v(t)|\, dt$ is the distance in feet that the car travels from $t = 30$ sec to $t = 60$ sec.

Trapezoidal approximation for $\int_{30}^{60} |v(t)|\, dt$:

$A = \frac{1}{2}(14 + 10)5 + \frac{1}{2}(10)(15) + \frac{1}{2}(10)(10) = 185$ ft

(b) $\int_0^{30} a(t)\, dt$ is the car's change in velocity in ft/sec from $t = 0$ sec to $t = 30$ sec.

$\int_0^{30} a(t)\, dt = \int_0^{30} v'(t)\, dt = v(30) - v(0)$

$= -14 - (-20) = 6$ ft/sec

(c) Yes. Since $v(35) = -10 < -5 < 0 = v(50)$, the IVT guarantees a t in $(35, 50)$ so that $v(t) = -5$.

(d) Yes. Since $v(0) = v(25)$, the MVT guarantees a t in $(0, 25)$ so that $a(t) = v'(t) = 0$.

Units of ft in (a) and ft/sec in (b)

16.

(a) Difference quotient; e.g.

$$W'(12) \approx \frac{W(15)-W(12)}{15-12} = -\frac{1}{3}\ ^\circ\text{C/day or}$$

$$W'(12) \approx \frac{W(12)-W(9)}{12-9} = -\frac{2}{3}\ ^\circ\text{C/day or}$$

$$W'(12) \approx \frac{W(15)-W(9)}{15-9} = -\frac{1}{2}\ ^\circ\text{C/day}$$

(b) $\frac{3}{2}(20+2(31)+2(28)+2(24)+2(22)+21) = 376.5$

Average temperature $\approx \frac{1}{15}(376.5) = 25.1\,^\circ\text{C}$

(c) $P'(12) = 10e^{-t/3} - \frac{10}{3}te^{-t/3}\Big|_{t=12}$

$= -30e^{-4} = -0.549\ ^\circ\text{C/day}$

This means that the temperature is decreasing at the rate of 0.549 °C/day when $t = 12$ days.

(d) $\frac{1}{15}\int_0^{15}\left(20 + 10te^{-t/3}\right)dt = 25.757\,^\circ\text{C}$

17.

(a) No; the amount of water is not increasing at $t = 15$ since $W(15) - R(15) = -121.09 < 0$.

(b) $1200 + \int_0^{18}(W(t) - R(t))\,dt = 1309.788$

1310 gallons

(c) $W(t) - R(t) = 0$

$t = 0, 6.4948, 12.9748$

t (hours)	gallons of water
0	1200
6.495	525
12.975	1697
18	1310

The values at the endpoints and the critical points show that the absolute minimum occurs when $t = 6.494$ or 6.495.

(d) $\int_{18}^{k} R(t)\,dt = 1310$

18.

(a) Since $R(6) = 4.438 > 0$, the number of mosquitoes is increasing at $t = 6$.

(b) $R'(6) = -1.913$

Since $R'(6) < 0$, the number of mosquitoes is increasing at a decreasing rate at $t = 6$.

(c) $1000 + \int_0^{31} R(t)\,dt = 964.335$

To the nearest whole number, there are 964 mosquitoes.

(d) $R(t) = 0$ when $t = 0$, $t = 2.5\pi$, or $t = 7.5\pi$

$R(t) > 0$ on $0 < t < 2.5\pi$

$R(t) < 0$ on $2.5\pi < t < 7.5\pi$

$R(t) > 0$ on $7.5\pi < t < 31$

The absolute maximum number of mosquitoes occurs at $t = 2.5\pi$ or at $t = 31$.

$1000 + \int_0^{2.5\pi} R(t)\,dt = 1039.357,$

There are 964 mosquitoes at $t = 31$, so the maximum number of mosquitoes is 1039, to the nearest whole number.

19.

(a) $P'(9) = 1 - 3e^{-0.6} = -0.646 < 0$
so the amount is not increasing at this time.

(b) $P'(t) = 1 - 3e^{-0.2\sqrt{t}} = 0$
$t = (5\ln 3)^2 = 30.174$
$P'(t)$ is negative for $0 < t < (5\ln 3)^2$ and positive for $t > (5\ln 3)^2$. Therefore there is a minimum at $t = (5\ln 3)^2$.

(c) $P(30.174) = 50 + \int_0^{30.174} \left(1 - 3e^{-0.2\sqrt{t}}\right) dt$
$= 35.104 < 40$, so the lake is safe.

(d) $P'(0) = 1 - 3 = -2$. The lake will become safe when the amount decreases by 10. A linear model predicts this will happen when $t = 5$.

20.

(a) $\int_0^{12} H(t)\,dt = 70.570$ or 70.571

(b) $H(6) - R(6) = -2.924$,
so the level of heating oil is falling at $t = 6$.

(c) $125 + \int_0^{12} (H(t) - R(t))\,dt = 122.025$ or 122.026

(d) The absolute minimum occurs at a critical point or an endpoint.
$H(t) - R(t) = 0$ when $t = 4.790$ and $t = 11.318$.

The volume increases until $t = 4.790$, then decreases until $t = 11.318$, then increases, so the absolute minimum will be at $t = 0$ or at $t = 11.318$.

$125 + \int_0^{11.318} (H(t) - R(t))\,dt = 120.738$