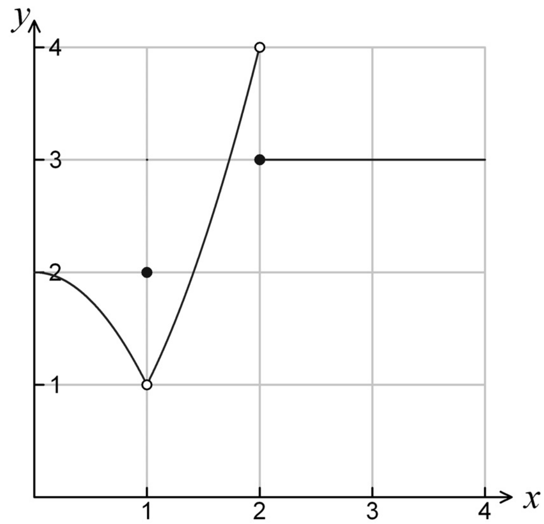
1. The function f and its graph are shown below:

$$f(x)=\begin{cases} -x^2+2 & 0\le x<1 \\ 2 & x=1 \\ x^2 & 1<x<2 \\ 3 & x\ge 2 \end{cases}$$



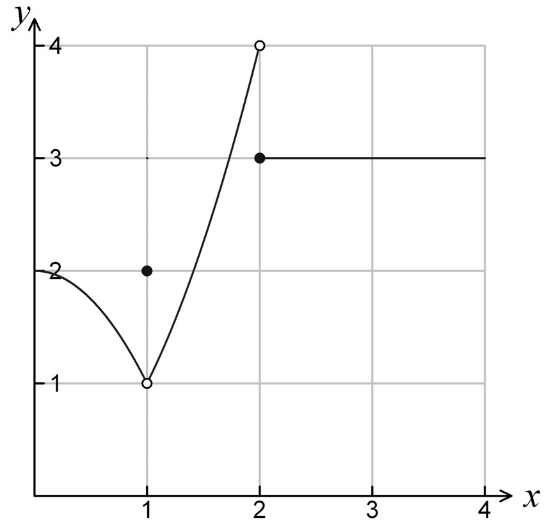
(a) Calculate $\lim_{x\to 2^-} f(x)$

(b) Which value is greater $\lim_{x\to 1} f(x)$ or $f(1)$? Justify your conclusion.

(c) At what value(s) of c on the interval $[0,4]$ does $\lim_{x\to c} f(x)$ not exist? Justify your conclusion.

1. The function f and its graph are shown below:

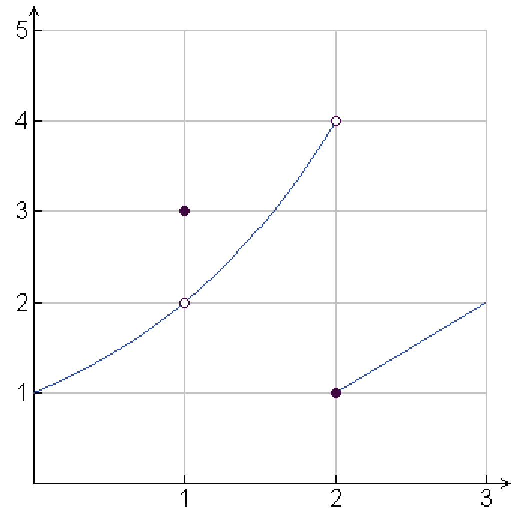
$$f(x)=\begin{cases}-x^2+2 & 0\le x<1\\ 2 & x=1\\ x^2 & 1<x<2\\ 3 & x\ge 2\end{cases}$$



(a) Calculate $\lim_{x\to 2^-} f(x)$ As we approach $x=2$ from the left, the graph of $f(x)$ nears 4 so $\lim_{x\to 2^-} f(x)=4$.	+1 $\lim_{x\to 2^-} f(x)=4$
(b) Which value is greater $\lim_{x\to 1} f(x)$ or $f(1)$? Justify your conclusion. Since $\lim_{x\to 1^-} f(x)=1$ and $\lim_{x\to 1^+} f(x)=1$, $\lim_{x\to 1} f(x)=1$. From the function equation and graph, we see $f(1)=2$. Therefore, $f(1)>\lim_{x\to 1} f(x)$.	+1 $\lim_{x\to 1} f(x)=1$ +1 $f(1)=2$ +2 $f(1)>\lim_{x\to 1} f(x)$
(c) At what value(s) of c on the interval $[0,4]$ does $\lim_{x\to c} f(x)$ not exist? Justify your conclusion. If the function f is continuous at $x=c$, then $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x)=f(c)$. We find the discontinuities in the graph of $f(x)$. Discontinuities occur at $x=1$ and $x=2$. Consider $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$. In part (b), we showed $\lim_{x\to 1} f(x)=1$ so the limit exists at $x=1$. Consider $\lim_{x\to 2} f(x)$. $\lim_{x\to 2^-} f(x)=4$ and $\lim_{x\to 2^+} f(x)=3$. Since the left and right hand limits are not equal, $\lim_{x\to 2} f(x)$ does not exist.	+1 If f is continuous at $x=c$, then $\lim_{x\to c} f(x)$ exists. +1 $\lim_{x\to 1} f(x)=1$ +1 $\lim_{x\to 2^-} f(x)=4$ and $\lim_{x\to 2^+} f(x)=3$ +1 At $x=2$, $\lim_{x\to c} f(x)$ does not exist.

2. The function f and its graph are shown below:

$$f(x)=\begin{cases} 2^x & 0\le x<1 \\ 3 & x=1 \\ 2^x & 1<x<2 \\ x-1 & 2\le x \end{cases}$$



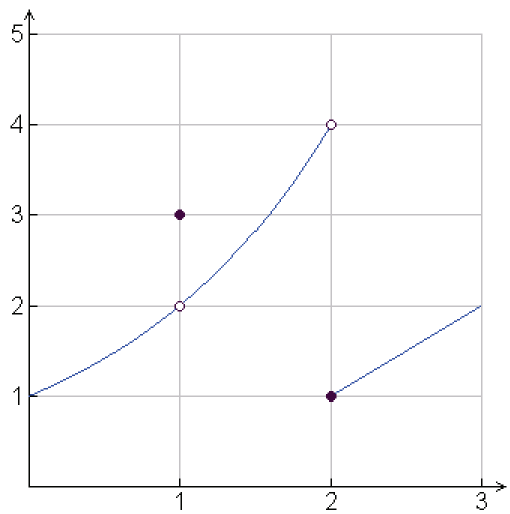
(a) Calculate $\lim_{x\to 2^+} f(x)$

(b) Which value is greater $\lim_{x\to 1} f(x)$ or $f(2)$? Justify your conclusion.

(c) At what value(s) of c on the interval $[0,3]$ does $\lim_{x\to c} f(x)$ not exist? Justify your conclusion.

2. The function f and its graph are shown below:

$$f(x)=\begin{cases} 2^x & 0\le x<1 \\ 3 & x=1 \\ 2^x & 1<x<2 \\ x-1 & 2\le x \end{cases}$$



(a) Calculate $\lim\limits_{x\to 2^+} f(x)$ As we approach $x=2$ from the right, the graph of $f(x)$ nears 1 so $\lim\limits_{x\to 2^+} f(x)=1$.	+1 $\lim\limits_{x\to 2^+} f(x)=1$
(b) Which value is greater $\lim\limits_{x\to 1} f(x)$ or $f(2)$? Justify your conclusion. Since $\lim\limits_{x\to 1^-} f(x)=2$ and $\lim\limits_{x\to 1^+} f(x)=2$, $\lim\limits_{x\to 1} f(x)=2$. From the function equation and graph, we see $f(2)=1$. Therefore, $\lim\limits_{x\to 1} f(x)>f(2)$.	+1 $\lim\limits_{x\to 1} f(x)=2$ +1 $f(2)=1$ +2 $\lim\limits_{x\to 1} f(x)>f(2)$
(c) At what value(s) of c on the interval $[0,3]$ does $\lim\limits_{x\to c} f(x)$ not exist? Justify your conclusion. If the function f is continuous at $x=c$, then $\lim\limits_{x\to c} f(x)$ exists and $\lim\limits_{x\to c} f(x)=f(c)$. We find the discontinuities in the graph of $f(x)$. Discontinuities occur at $x=1$ and $x=2$. Consider $\lim\limits_{x\to 1} f(x)$ and $\lim\limits_{x\to 2} f(x)$. In part (b), we showed $\lim\limits_{x\to 1} f(x)=2$ so the limit exists at $x=1$. Consider $\lim\limits_{x\to 2} f(x)$. $\lim\limits_{x\to 2^-} f(x)=4$ and $\lim\limits_{x\to 2^+} f(x)=1$. Since the left and right hand limits are not equal, $\lim\limits_{x\to 2} f(x)$ does not exist.	+1 If f is continuous at $x=c$, then $\lim\limits_{x\to c} f(x)$ exists. +1 $\lim\limits_{x\to 1} f(x)=2$ +1 $\lim\limits_{x\to 2^-} f(x)=4$ and $\lim\limits_{x\to 2^+} f(x)=1$ +1 At $x=2$, $\lim\limits_{x\to c} f(x)$ does not exist.

3. The function g is defined as follows: $g(x)=\dfrac{x^2-4}{x^2+5x+6}$.

(a) Use a table of values to estimate $\lim_{x\to -2} g(x)$ accurate to three decimal places. Show the work that leads to your conclusion.

(b) Calculate $\lim_{x\to -3^-} g(x)$ and $\lim_{x\to -3^+} g(x)$? Show the work that leads to your conclusion.

(c) $g(x)$ is undefined at $x=-2$ and $x=-3$. Explain why $\lim_{x\to -2} g(x)$ exists but $\lim_{x\to -3} g(x)$ does not exist.

3. The function g is defined as follows: $g(x) = \dfrac{x^2 - 4}{x^2 + 5x + 6}$.

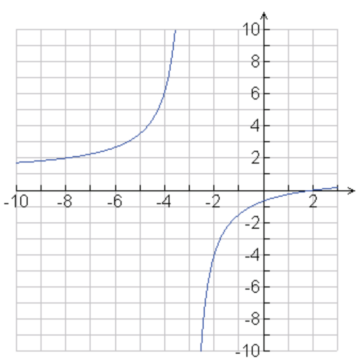
(a) Use a table of values to estimate $\lim_{x \to -2} g(x)$ accurate to three decimal places. Show the work that leads to your conclusion.

x	$g(x)$
−2.01	−4.051
−2.001	−4.005
−2.00001	−4.000
−2	undef
−1.99999	−4.000
−1.999	−3.995
−1.99	−3.950

$\lim_{x \to -2} g(x) = -4.000$

+1 $\lim_{x \to -2} g(x) = -4.000$
+1 Table of values is used
+1 Table of values shows two table entries on either side of −2 with $g(x) = -4.000$.

(b) Calculate $\lim_{x \to -3^-} g(x)$ and $\lim_{x \to -3^+} g(x)$? Show the work that leads to your conclusion.



x	$g(x)$
−3.01	501
−3.001	5001
−3.0001	50,001
−3	undef
−2.9999	−49,999
−2.999	−4999
−2.99	−499

$\lim_{x \to -3^-} g(x) = \infty$

$\lim_{x \to -3^+} g(x) = -\infty$

+1 $\lim_{x \to -3^-} g(x) = \infty$
+1 $\lim_{x \to -3^+} g(x) = -\infty$
+2 Table of values or graph support conclusion

(c) $g(x)$ is undefined at $x = -2$ and $x = -3$. Explain why $\lim_{x \to -2} g(x)$ exists but $\lim_{x \to -3} g(x)$ does not exist.

The function $g(x) = \dfrac{x^2 - 4}{x^2 + 5x + 6}$ may be rewritten as $g(x) = \dfrac{(x+2)(x-2)}{(x+2)(x+3)}$. Observe that $x \neq -2$ and $x \neq -3$ because these values make the denominator zero. With this restriction, $g(x)$ may be further simplified to $g(x) = \dfrac{x-2}{x+3}$. This function has a vertical asymptote at $x = -3$ and a removable discontinuity at $x = -2$. Therefore, $\lim_{x \to -2} g(x)$ exists but $\lim_{x \to -3} g(x)$ does not exist.

+1 Explains that $g(x)$ has a vertical asymptote at $x = -3$.

+1 Explains that $g(x)$ has a removable discontinuity at $x = -2$.

4. The function g is defined as follows: $g(x) = \dfrac{x^2 - 2x}{x^2 - 4}$

(a) Use a table of values to estimate $\lim_{x \to 2} g(x)$ accurate to three decimal places. Show the work that leads to your conclusion.

(b) Calculate $\lim_{x \to -2^+} g(x)$ and $\lim_{x \to -2^-} g(x)$? Show the work that leads to your conclusion.

(c) $g(x)$ is undefined at $x = -2$ and $x = 2$. Explain why $\lim_{x \to 2} g(x)$ exists but $\lim_{x \to -2} g(x)$ does not exist.

4. The function g is defined as follows: $g(x)=\dfrac{x^2-2x}{x^2-4}$.

<table>
<tr>
<td>

(a) Use a table of values to estimate $\lim\limits_{x\to 2} g(x)$ accurate to three decimal places. Show the work that leads to your conclusion.

<table>
<tr><th>x</th><th>$g(x)$</th></tr>
<tr><td>1.9</td><td>0.487</td></tr>
<tr><td>1.99</td><td>0.499</td></tr>
<tr><td>1.999</td><td>0.500</td></tr>
<tr><td>2</td><td>undef</td></tr>
<tr><td>2.001</td><td>0.500</td></tr>
<tr><td>2.01</td><td>0.501</td></tr>
<tr><td>2.1</td><td>0.512</td></tr>
</table>

$\lim\limits_{x\to 2} g(x)=0.500$

</td>
<td>

+1 $\lim\limits_{x\to 2} g(x)=0.500$
+1 Table of values is used
+1 Table of values shows two table entries on either side of 2 with $g(x)=0.500$.

</td>
</tr>
<tr>
<td>

(b) Calculate $\lim\limits_{x\to -2^+} g(x)$ and $\lim\limits_{x\to -2^-} g(x)$? Show the work that leads to your conclusion.

<table>
<tr><th>x</th><th>$g(x)$</th></tr>
<tr><td>–2.1</td><td>21</td></tr>
<tr><td>–2.01</td><td>201</td></tr>
<tr><td>–2.001</td><td>2001</td></tr>
<tr><td>–2</td><td>undef</td></tr>
<tr><td>–1.999</td><td>–1999</td></tr>
<tr><td>–1.99</td><td>–199</td></tr>
<tr><td>–1.9</td><td>–19</td></tr>
</table>

$\lim\limits_{x\to -2^+} g(x)=-\infty$

$\lim\limits_{x\to -2^-} g(x)=\infty$

</td>
<td>

+1 $\lim\limits_{x\to -2^+} g(x)=-\infty$
+1 $\lim\limits_{x\to -2^-} g(x)=\infty$
+2 Table of values or graph support conclusion

</td>
</tr>
<tr>
<td>

(c) $g(x)$ is undefined at $x=-2$ and $x=2$. Explain why $\lim\limits_{x\to 2} g(x)$ exists but $\lim\limits_{x\to -2} g(x)$ does not exist.

The function $g(x)=\dfrac{x^2-2x}{x^2-4}$ may be rewritten as $g(x)=\dfrac{x(x-2)}{(x+2)(x-2)}$. Observe that $x\neq -2$ and $x\neq 2$ because these values make the denominator zero. With this restriction, $g(x)$ may be further simplified to $g(x)=\dfrac{x}{x+2}$. This function has a vertical asymptote at $x=-2$ and a removable discontinuity at $x=2$. Therefore, $\lim\limits_{x\to 2} g(x)$ exists but $\lim\limits_{x\to -2} g(x)$ does not exist.

</td>
<td>

+1 Explains that $g(x)$ has a vertical asymptote at $x=-2$.

+1 Explains that $g(x)$ has a removable discontinuity at $x=2$.

</td>
</tr>
</table>

5. Define $f(x)=\begin{cases} x-4 & 1\le x<2 \\ \dfrac{1}{x-3} & 2\le x<5 \\ -x+5.5 & 5\le x \end{cases}$

(a) Show $f(x)$ is continuous at $x=5$.

(b) Where on the interval $[1,4]$ is f discontinuous? Show the work that leads to your conclusion.

(c) Explain the difference between a removable and an irremovable discontinuity.

5. Define $f(x)=\begin{cases} x-4 & 1\le x<2 \\ \dfrac{1}{x-3} & 2\le x<5 \\ -x+5.5 & 5\le x \end{cases}$

(a) Show $f(x)$ is continuous at $x=5$. 1. $\lim_{x\to 5^+} f(x)=\lim_{x\to 5^-} f(x)=0.5$ so $\lim_{x\to 5} f(x)$ exists and is equal to 0.5 2. $f(5)=0.5$ 3. $\lim_{x\to 5} f(x)=f(5)$	+1 Show $\lim_{x\to 5} f(x)$ exists +1 $f(5)=0.5$ +1 $\lim_{x\to 5} f(x)=f(5)$
(b) Where on the interval $[1,4]$ is f discontinuous? Show the work that leads to your conclusion. Since linear functions are continuous, f is continuous on $[1,2)$. Since $\lim_{x\to 2^-} f(x)=-2$ and $\lim_{x\to 2^+} f(x)=-1$, $\lim_{x\to 2} f(x)$ does not exist. Therefore, f is discontinuous at $x=2$. The function $\frac{1}{x-3}$ is defined for all values of x on $[2,4]$ except at $x=3$. Therefore, a discontinuity occurs at $x=3$. For every other value c in the interval $[2,4]$, $\lim_{x\to c} f(x)=f(c)$. Thus f is continuous at every other point.	+1 Show that $\lim_{x\to 2} f(x)$ does not exist. +1 States f is discontinuous at $x=2$ +1 State f is discontinuous at $x=3$ because f is undefined there +1 Provide justification that f is continuous everywhere else.
(c) Explain the difference between a removable and an irremovable discontinuity. If $\lim_{x\to c} f(x)$ exists but $\lim_{x\to c} f(x)\ne f(c)$, then a removable discontinuity occurs at $x=c$. If $\lim_{x\to c} f(x)$ does not exist then an irremovable discontinuity occurs at $x=c$.	+1 Concept of $\lim_{x\to c} f(x)$ exists but $\lim_{x\to c} f(x)\ne f(c)$ for removable +1 Concept of $\lim_{x\to c} f(x)$ does not exist for irremovable

6. Define $f(x)=\dfrac{\ln|x-1|}{x}$.

(a) Show $f(x)$ is continuous at $x=2$.

(b) Where on the interval $[-2,2]$ is f discontinuous? Show the work that leads to your conclusion.

(c) Classify the discontinuities in part (b) as removable or irremovable.

6. Define $f(x)=\dfrac{\ln|x-1|}{x}$.

(a) Show $f(x)$ is continuous at $x=2$.	
1. $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^-} f(x)=0$ so $\lim_{x\to 2} f(x)$ exists and is equal to 0. 2. $f(2)=\dfrac{\ln\lvert 2-1\rvert}{2}=\dfrac{\ln\lvert 1\rvert}{2}=0$ 3. $\lim_{x\to 2} f(x)=f(2)$	+1 Show $\lim_{x\to 2} f(x)$ exists +1 $f(2)=0$ +1 $\lim_{x\to 2} f(x)=f(2)$
(b) Where on the interval $[-2,2]$ is f discontinuous? Show the work that leads to your conclusion.	
We know $y=\ln x$ is continuous for positive values of x. Consequently, $y=\ln\lvert x\rvert$ is continuous for all nonzero values of x; however, it is undefined and thus discontinuous when $x=0$. Similarly, $y=\ln\lvert x-1\rvert$ is continuous for all values of x except at $x=1$, where it is undefined. Therefore, the function $f(x)=\dfrac{\ln\lvert x-1\rvert}{x}$ is discontinuous at $x=1$. It is also discontinuous at $x=0$ because f is undefined there.	+2 Shows that f is undefined at $x=0$ and $x=1$. +1 States that f is discontinuous where it is not defined. +1 Provides justification that f is continuous everywhere else.
(c) Classify the discontinuities in part (b) as removable or irremovable.	+1 $x=0$ is a removable discontinuity +1 $x=1$ is an irremovable discontinuity

x	$f(x)$
–0.1	–0.953
–0.01	–0.995
–0.001	–1.000
0	undef
0.001	–1.001
0.01	–1.005
0.1	–1.054

x	$f(x)$
0.99	–4.652
0.999	–6.915
0.9999	–9.211
1	undef
1.0001	–9.209
1.001	–6.901
1.01	–4.560

In the table on the left, we see that $\lim_{x\to 0} f(x)=-1$. Since the limit exists, $x=0$ is a removable discontinuity. In the table on the right, it initially appears as if $\lim_{x\to 1} f(x)=-9.209$; however, we notice that the values of f are changing rather dramatically as we near $x=1$. In fact, $f(1.0000001)=-16.118$ which is not at all close to –9.209. Consequently, we conclude that $\lim_{x\to 1} f(x)=-\infty$. Therefore, $x=1$ is an irremovable discontinuity.

7. Define $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 1}$.

(a) Find $\lim_{x \to -1} f(x)$ analytically.

(b) Define $g(x) = \dfrac{x+1}{x-1}$. What is the relationship between f and g?

(c) Explain why $\lim_{x \to 1^+} f(x) = \infty$.

7. Define $f(x)=\dfrac{x^2+2x+1}{x^2-1}$.

(a) Find $\lim\limits_{x\to -1} f(x)$ analytically. $\lim\limits_{x\to -1} f(x)=\lim\limits_{x\to -1}\dfrac{x^2+2x+1}{x^2-1}$ $=\lim\limits_{x\to -1}\dfrac{(x+1)^2}{(x-1)(x+1)}$ $=\lim\limits_{x\to -1}\dfrac{(x+1)}{(x-1)}$ $=\dfrac{-1+1}{-1-1}$ $=0$	+1 Correctly factor numerator and denominator. +1 Cancel $x+1$ +2 $\lim\limits_{x\to -1} f(x)=0$
(b) Define $g(x)=\dfrac{x+1}{x-1}$. What is the relationship between f and g? The graphs of f and g will look essentially the same. However, the graph of f will have a removable discontinuity at $x=-1$. or The domain of f is all real numbers except $x=\pm 1$ whereas the domain of g is all real numbers except $x=1$.	+2 Observe that the graphs of f and g differ by a single point OR that the domains of f and g differ by a single value
(c) Explain why $\lim\limits_{x\to 1^+} f(x)=\infty$. We evaluate $f(x)$ at values of x increasingly close to $x=1$ as we approach $x=1$ from the right. $f(1.1)=21$ $f(1.01)=201$ $f(1.001)=2001$ $f(1.0001)=20001$ We see that the values of f become increasingly large as x nears $x=1$ from the right. Thus $\lim\limits_{x\to 1^+} f(x)=\infty$	+1 Evaluate f at four or more values of x within $[1,1.1]$ +2 Show a trend of increasing function values as x nears 1 from the right.

8. Define $f(x) = \dfrac{\cos x - 1}{\sin x}$.

(a) Use a table of values to estimate $\lim_{x \to 0} f(x)$ accurate to three decimal places.

(b) Define $g(x) = \dfrac{x+1}{f(x)+2}$. Use properties of limits to show that $\lim_{x \to 0} g(x) = \dfrac{1}{2}$.

(c) Is g continuous? Show the work that leads to your conclusion.

8. Define $f(x)=\dfrac{\cos x-1}{\sin x}$.

(a) Use a table of values to estimate $\lim\limits_{x\to 0} f(x)$ accurate to three decimal places.

x	$f(x)$
–0.01	0.00500
–0.001	0.00050
–0.0001	0.00005
0	undef
0.0001	–0.00005
0.001	–0.00050
0.01	–0.005

$\lim\limits_{x\to 0} f(x)=0$

+1 $\lim\limits_{x\to 0} f(x)=0$
+1 Table of values is used
+1 Table of values shows two table entries on either side of 0 with $f(x)=0.000$.

(b) Define $g(x)=\dfrac{x+1}{f(x)+2}$. Use properties of limits to show that $\lim\limits_{x\to 0} g(x)=\dfrac{1}{2}$.

In part (a), we showed that $\lim\limits_{x\to 0} f(x)=0$. We also know that $\lim\limits_{x\to 0}(x+1)=1$. So

$$\lim_{x\to 0} g(x)=\frac{\lim\limits_{x\to 0}(x+1)}{\lim\limits_{x\to 0} f(x)+2}=\frac{1}{0+2}=\frac{1}{2}$$

+1 $\lim\limits_{x\to 0}(x+1)=1$
+1 $\lim\limits_{x\to 0} f(x)+2=2$
+1 properly apply limit properties to show $\lim\limits_{x\to 0} g(x)=\dfrac{1}{2}$

(c) Is g continuous? Show the work that leads to your conclusion.

$$g(x)=\frac{x+1}{f(x)+2}=\frac{x+1}{\frac{\cos x-1}{\sin x}+2}$$

At $x=0$, $\sin x=0$. This makes the denominator undefined. Therefore, g is not continuous at $x=0$

+1 Find a discontinuity
+2 State that function is not continuous.

9. Define $f(x)=\dfrac{1-\cos x}{x}$ and $g(x)=\dfrac{2}{x}$ on the interval $(0,\infty)$.

(a) Show $g(x)\geq f(x)\geq 0$

(b) Identify all vertical asymptotes of f or g. Show the work that leads to your conclusion.

(c) Determine $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$. Show the work that leads to your conclusion.

9. Define $f(x) = \frac{1 - \cos x}{x}$ and $g(x) = \frac{2}{x}$ on the interval $(0, \infty)$.

(a) Show $g(x) \geq f(x) \geq 0$.	

1 cos 1

1 cos 1

2 1 cos 0

2 1 cos