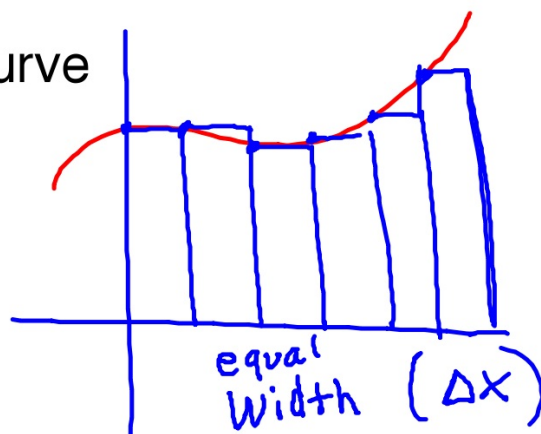


## Left, Right, Midpoint and Trapezoidal Sums

Approximating area under a curve

$$\int_a^b f(x) dx$$



$n$ : number of rectangles  
 $n \rightarrow \infty$ ;  $\Delta x \rightarrow 0$

If there is a constant width, the width can be calculated by:

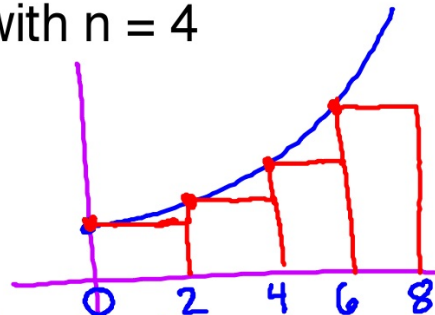
$$\text{Width} = (b - a)/n$$

$$\left[ \begin{array}{c} 0, 12 \\ a \quad b \end{array} \right] n = 4$$

$$W = \frac{12 - 0}{4} = 3$$

Left sum: Start at the LEFT of the interval  $w = \frac{8-0}{4} = 2$

$f(x) = x^2 + 1$  on the interval  $[0, 8]$  with  $n = 4$



$$\begin{aligned} \int_0^8 f(x) dx &\approx 2 \cdot f(0) + 2 \cdot f(2) + 2 \cdot f(4) + 2 \cdot f(6) \\ &\approx 2(1 + 5 + 17 + 37) \\ &\approx 120 \end{aligned}$$

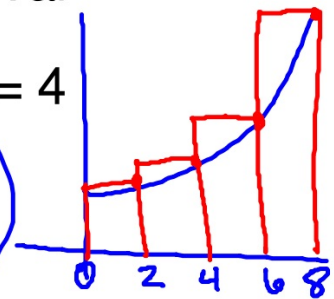
Right Sum: start at the right of the interval

$f(x) = x^2 + 1$  for the interval  $[0, 8]$  with  $n = 4$

$$\int_0^8 f(x) dx \approx 2(f(8) + f(6) + f(4) + f(2))$$

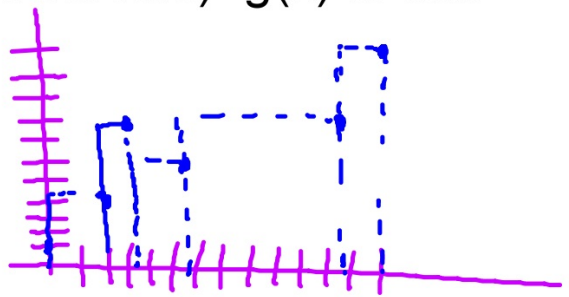
$$\approx 2(65 + 37 + 17 + 5)$$

$$\approx 248$$



Right sum with a chart (5 sub-intervals)  $g(x)$  is diff.

$x$	0	2	3	7	12	13
$f(x)$	1	4	8	5	8	11

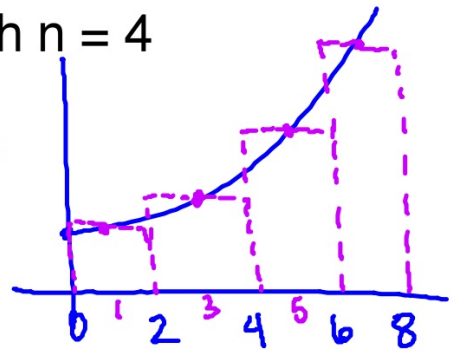


$$\int_0^{13} f(x) dx \approx 1f(13) + 5f(12) + 4f(7) + 1f(3) + 2f(2)$$
$$1 \cdot 11 + 5 \cdot 8 + 4 \cdot 5 + 1 \cdot 8 + 2 \cdot 4$$
$$\approx 87$$

Midpoint Sums: Find the midpoints of the sub-intervals

$f(x) = x^2 + 1$  on the interval  $[0, 8]$  with  $n = 4$

$$\int_0^8 f(x) dx \approx 2(f(1) + f(3) + f(5) + f(7))$$
$$\approx 2(2 + 10 + 26 + 50)$$
$$\approx 176$$

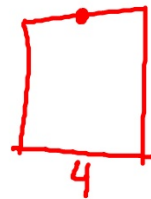


Midpoint with a chart (3 sub-intervals)

X	0	4	8	12	16	20	24
g(x)	1	5	7	8	10	9	13

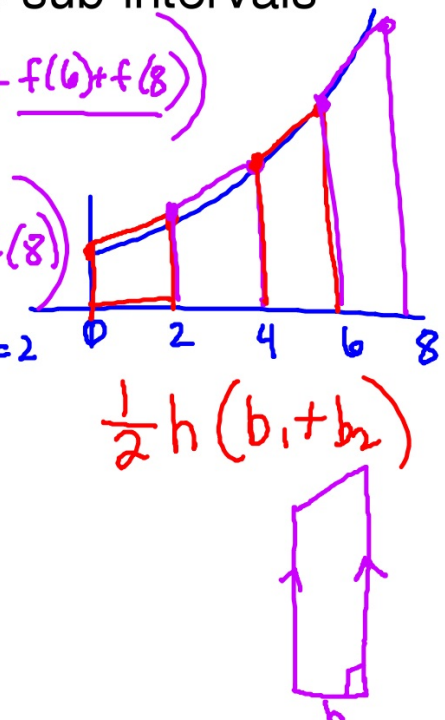
$$W = \frac{b-a}{n}$$
$$W = \frac{24-0}{3}$$
$$W = 8$$

$$\int_0^{24} g(x) dx \approx 8 (g(4) + g(12) + g(20))$$
$$\approx 8 (5 + 8 + 9)$$
$$\approx 176$$



Trapezoids: Approximating the area using trapezoids

$f(x) = x^2 + 1$  on the interval  $[0, 8]$  with 4 sub-intervals

$$\int_0^8 f(x) dx \approx \frac{1}{2} \cdot 2 \left( \underbrace{f(0)} + \underbrace{f(2)} + \underbrace{f(2)} + \underbrace{f(4)} + \underbrace{f(4)} + \underbrace{f(6)} + \underbrace{f(6)} + \underbrace{f(8)} \right)$$
$$\approx \frac{1}{2} \left( f(0) + 2f(2) + 2f(4) + 2f(6) + f(8) \right)$$
$$\approx \frac{1}{2} \left( 1 + 2 \cdot 5 + 2 \cdot 17 + 2 \cdot 37 + 65 \right)$$
$$\approx 184$$


$\frac{1}{2} h (b_1 + b_2)$



Trapezoids with a chart (3 sub-intervals)

$x$	1	3	7	9
$g(x)$	5	8	4	3

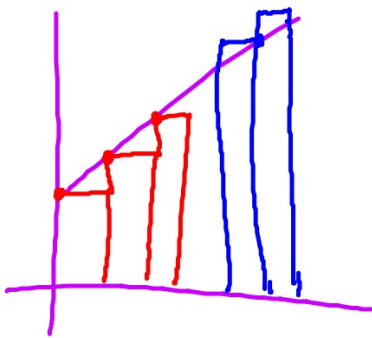
$$\frac{1}{2} h (b_1 + b_2)$$



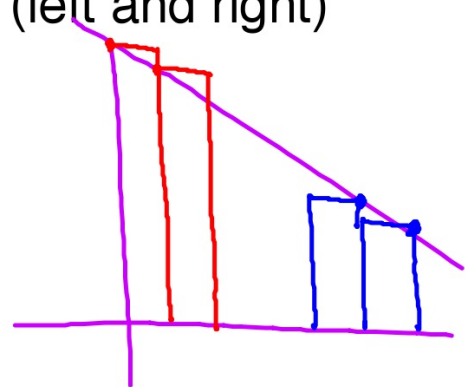
$$\int_1^9 g(x) dx \approx \frac{1}{2} \left( 2(5+8) + \underline{4(8+4)} + \underline{2(4+3)} \right)$$
$$\approx 44$$

Will the estimate be an under or over approximation?

Increasing/Decreasing Functions (left and right)

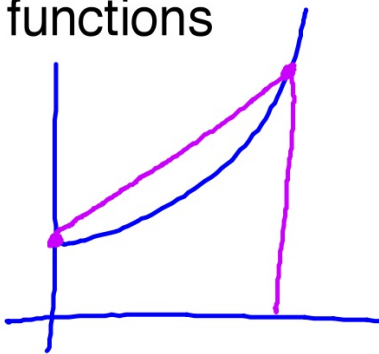


Increasing function  
Left Riemann: under  
Right Riemann: over

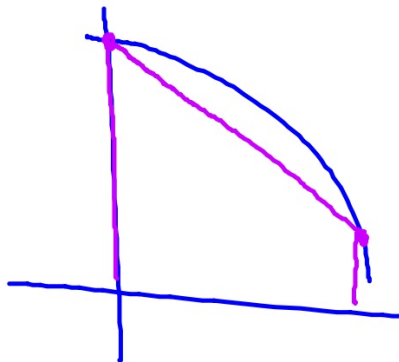


Decreasing function  
Left Riemann: over  
Right Riemann: under

Will the estimate be an under or over approximation for Concave Up/Concave Down functions

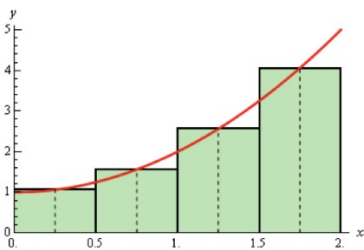


Concave up  
Over approx.  
Trapezoids



Concave down  
Under approx.  
Trapezoids

Will the estimate be an under or over approximation for Concave Up/Concave Down functions

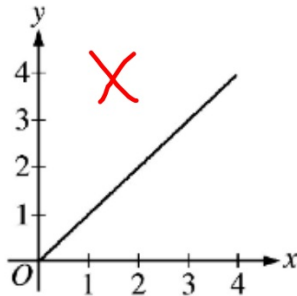
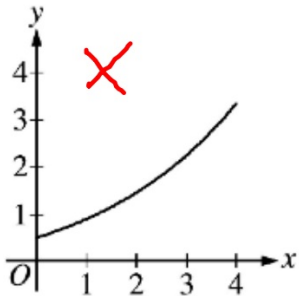
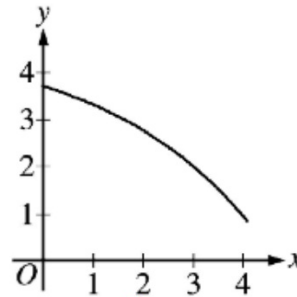
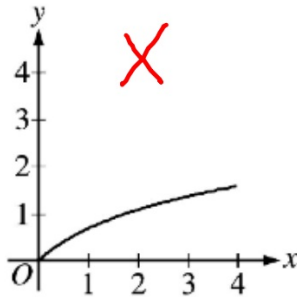
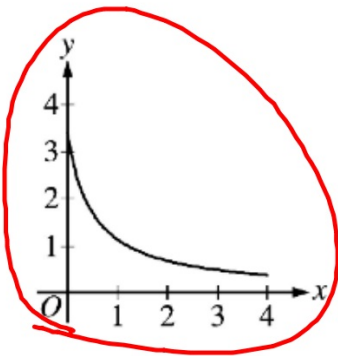


Concave up  
Under approx.  
Midpoint



Concave down  
Over approx.  
Midpoint

If a trapezoidal sum over approximates  $\int_0^4 f(x)dx$ , and a right Riemann sum under approximates  $\int_0^4 f(x)dx$ , which of the following could be the graph of  $y = f(x)$ ?



decr.

under approx trap.

**Comments:**

If the graph is decreasing, then  $\text{Right}(n) < \int_a^b f(x)dx < \text{Left}(n)$  for the right Riemann and left Riemann sums using  $n$  subintervals.

If the graph is concave up, then  $\text{Mid}(n) < \int_a^b f(x)dx < \text{Trap}(n)$  for the trapezoid sum and the midpoint Riemann sum using  $n$  subintervals.

Graph (A) is decreasing and concave up, and therefore could be the graph of  $y = f(x)$ .

If the graph is increasing or concave down, the respective inequalities are reversed.