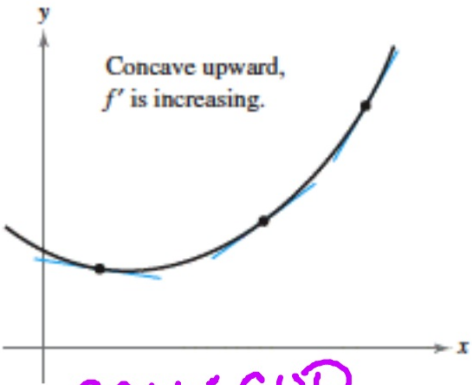
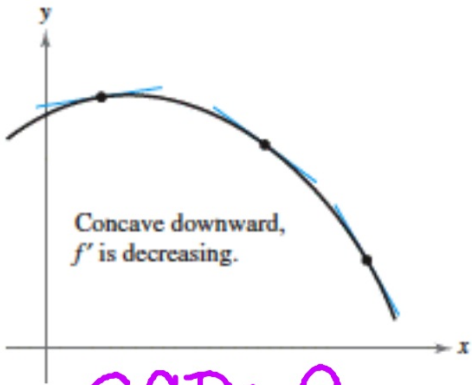
3.4 Concavity and the Second Derivative Test

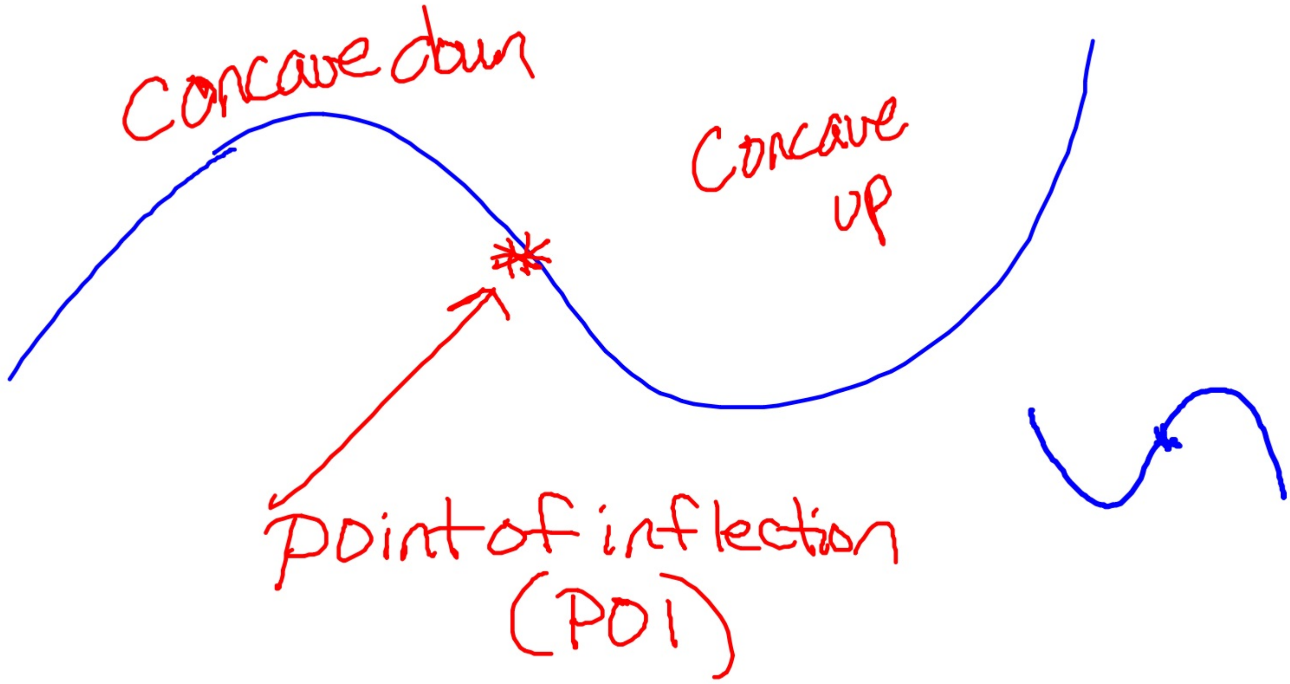
- Determine intervals on which a function is concave upward or concave downward.
- Find any points of inflection of the graph of a function.
- Apply the Second Derivative Test to find relative extrema of a function.



CCU: cup
$f'' > 0$

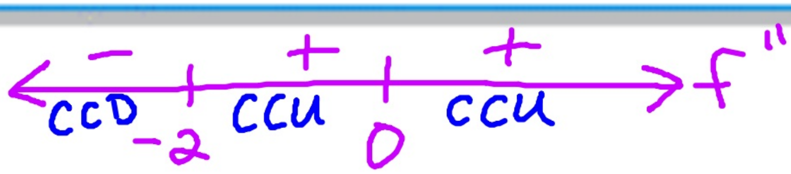


CCD: frown
$f'' < 0$


Concave down
Concave up
point of inflection
(POI)

EOREM 3.8 POINTS OF INFLECTION

$(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f''
es not exist at $x = c$.



POI at x = -2 because f'' changes signs at this point

CCD: (-inf, -2) because f'' < 0 on this interval

CCU: (-2,0) U (0,inf) because f'' > 0 on this interval

We use the first derivative to find:

intervals of increasing/decreasing and

relative extrema

We use the second derivative to find:

intervals of concavity and

points of inflection

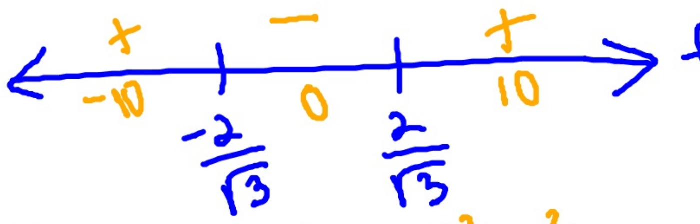
1. $f(x) = \frac{1}{4}x^4 - 2x^2$ D: $(-\infty, \infty)$

$f'(x) = x^3 - 4x$

$f''(x) = 3x^2 - 4$

$0 = 3x^2 - 4$

$\pm\frac{2}{\sqrt{3}} = x$



POI: $\left(\frac{-2}{\sqrt{3}}, \frac{-20}{9}\right)$ $\left(\frac{2}{\sqrt{3}}, \frac{-20}{9}\right)$

f is concave down $\left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ because f'' < 0 on this interval.

Concave up (-inf, $\frac{-2}{\sqrt{3}}$) and ($\frac{2}{\sqrt{3}}$,inf) because f'' >0 on these intervals

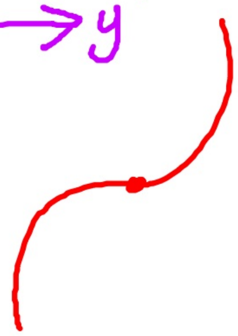
Points of inflection at x = $\pm\frac{2}{\sqrt{3}}$ because f'' changes signs at these values.

① $y = x^2$

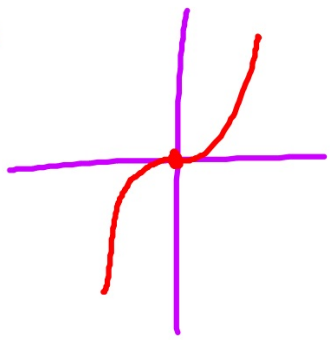
$y' = 2x$

$y'' = 2$

$$\begin{array}{c} + \\ \longleftrightarrow y'' \\ \text{CCU} \end{array}$$



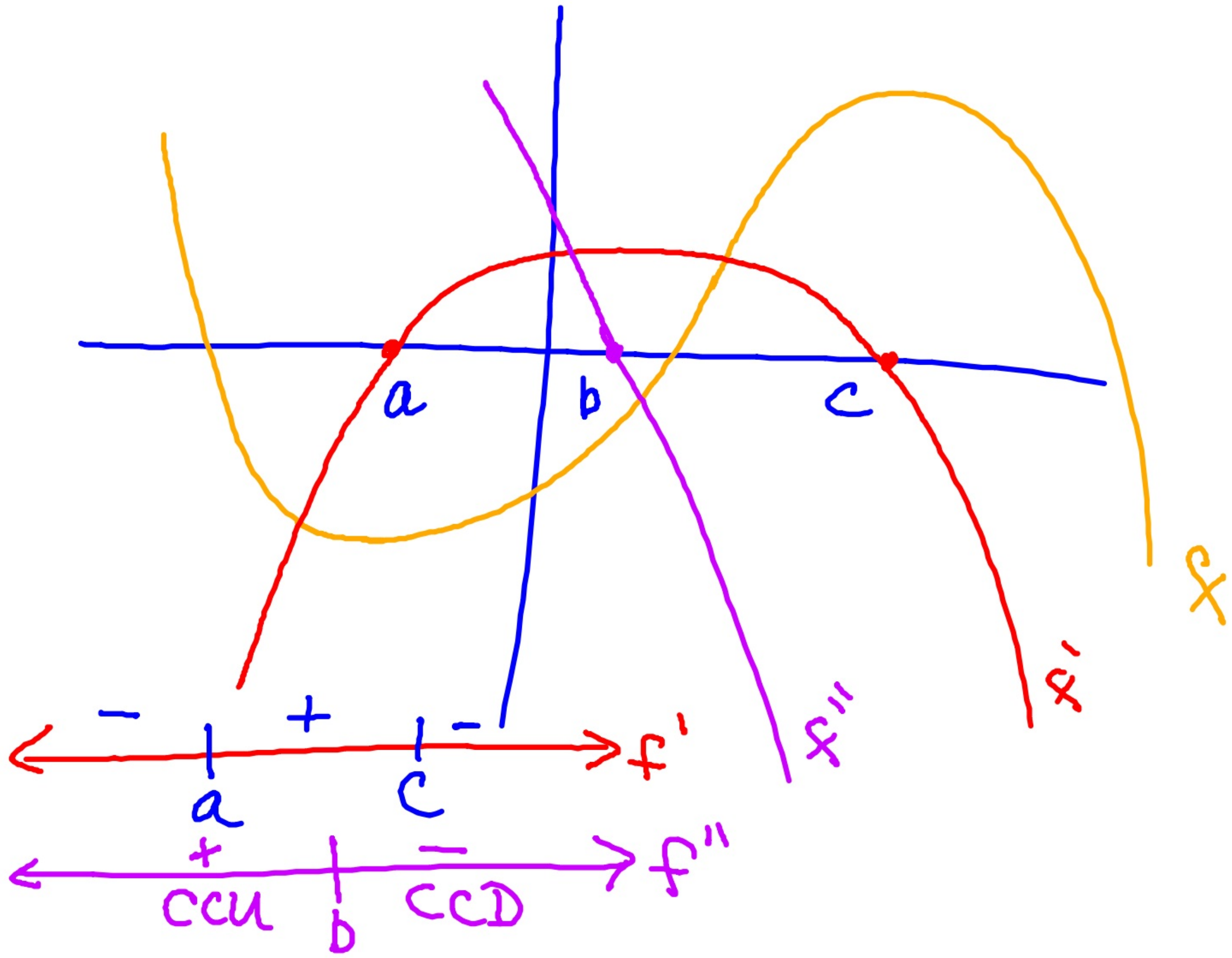
② $y = x^3$



$y' = 3x^2$

$y'' = 6x$

$$\begin{array}{ccc} - & & + \\ \longleftarrow & | & \longrightarrow y'' \\ \text{CCD} & 0 & \text{CCU} \end{array}$$



a
b
c
f
f'
f''
−
+
−
a
c
f'
+
−
CCU
b
CCD
f''

27.) $f(x) = x\sqrt{x-3}$ find POI $D: \{x \mid x \geq 3\}$

$$f'(x) = x \cdot \frac{1}{2}(x-3)^{-1/2} + (x-3)^{1/2} \cdot 1$$

$$= \frac{x}{2\sqrt{x-3}} + \frac{\sqrt{x-3}}{1}$$

$$= \frac{x + 2(x-3)}{2\sqrt{x-3}}$$

$$= \frac{3x-6}{2\sqrt{x-3}} = \frac{3(x-2)}{2\sqrt{x-3}}$$

3 [+ → f'

f is always increasing

$$f'(x) = \frac{3}{2}\left[(x-2)(x-3)^{-1/2}\right]$$

$$f''(x) = \frac{3}{2}\left((x-2)\left(-\frac{1}{2}(x-3)^{-3/2}\right) + (x-3)^{-1/2} \cdot 1\right)$$

$$= \frac{3}{2}\left[(x-3)^{-3/2}\left(-\frac{1}{2}(x-2) + (x-3)\right)\right]$$

$$= \frac{3}{2}\left[(x-3)^{-3/2}\left(\frac{1}{2}x - 2\right)\right]$$

$$= \frac{3\left(\frac{1}{2}x - 2\right)}{2(x-3)^{3/2}}$$

$x = 4$

3 [− | 4 + → f''

But, there are two tests that have an official name

First Derivative Test

Second Derivative Test

BOTH tests are ways to locate relative extrema

local

THEOREM 3.9 SECOND DERIVATIVE TEST

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Find the relative extrema using the second derivative test.

#1 $f(x) = -3x^5 + 5x^3$.