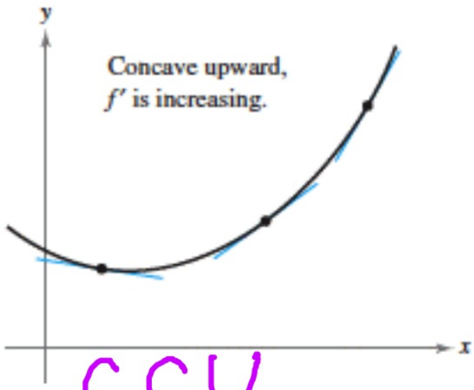
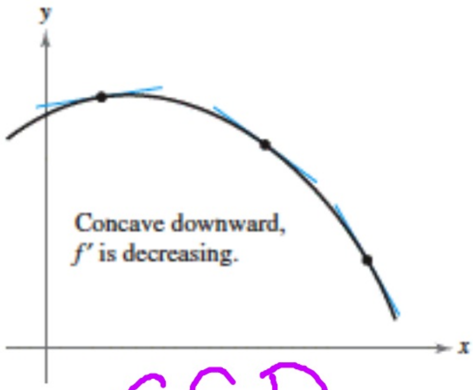
3.4 Concavity and the Second Derivative Test

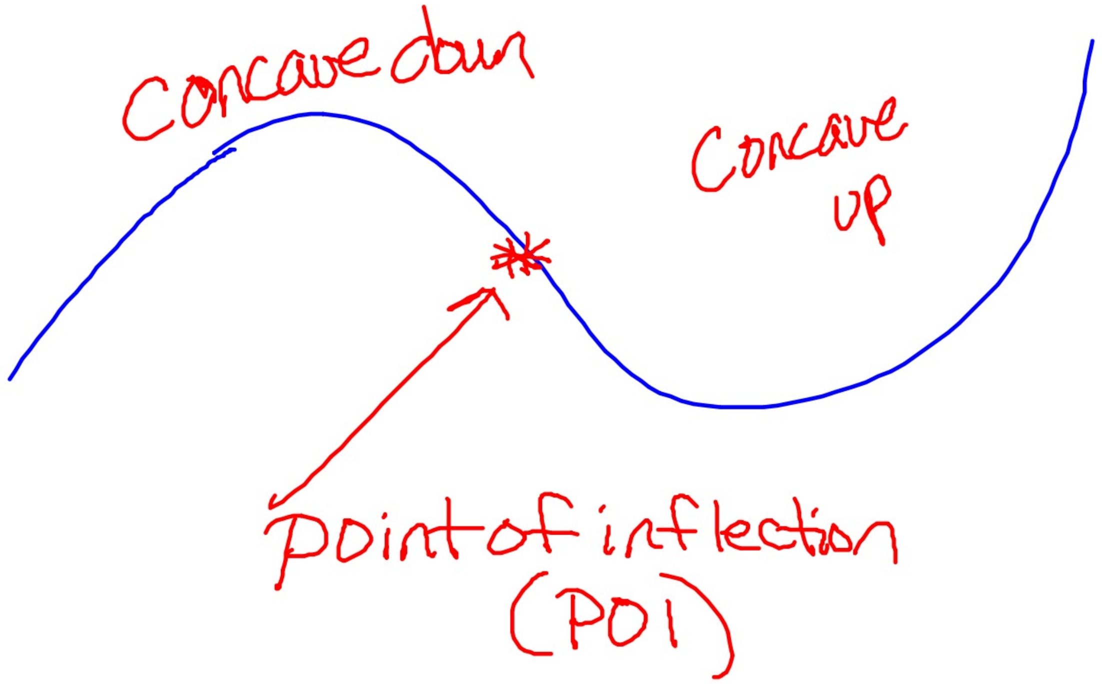
- Determine intervals on which a function is concave upward or concave downward.
- Find any points of inflection of the graph of a function.
- Apply the Second Derivative Test to find relative extrema of a function.



CCU
(cup)
+ (f'')

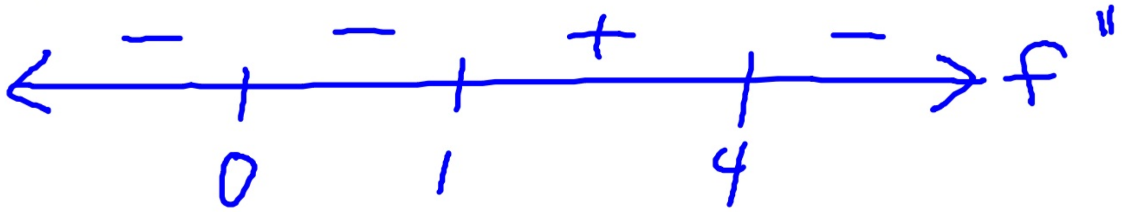


CCD
(frown)
− (f'')


concave down
concave up
point of inflection
(POI)

3.8 POINTS OF INFLECTION

is a point of inflection of the graph of f, then either $f''(c) = 0$ or
xist at $x = c$.



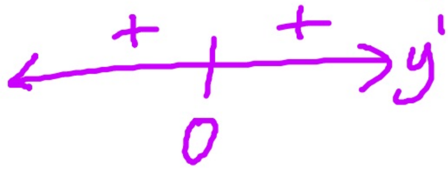
CCU $(1,4)$ $f'' > 0$

CCD $(-\infty,0) \cup (0,1) \cup (4,\infty)$ $f'' < 0$

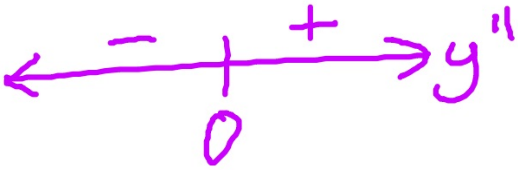
POI $x=1$ and $x=4$; f'' changes signs at these values

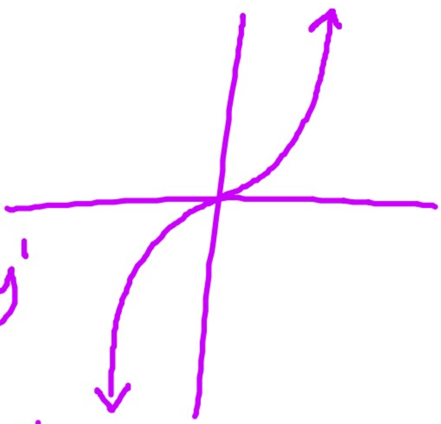
$y = x^3$

$y' = 3x^2$



$y'' = 6x$





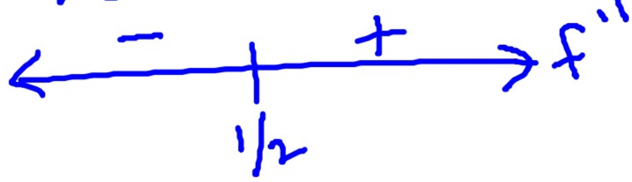
Find the intervals where f is concave up or down.
Also, find the points of inflection.

#1

$$f(x) = 2x^3 - 3x^2 - 12x + 5$$

$$f'(x) = 6x^2 - 6x - 12$$

$$f''(x) = 12x - 6$$



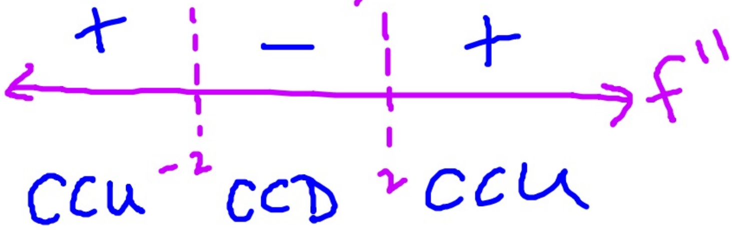
Concave down (-inf,1/2) because f''<0 on this interval
concave up (1/2,inf) because f''>0 on this interval
POI at (1/2,$-1\frac{1}{2}$) because f'' changes signs at x = 1/2

$$f\left(\tfrac{1}{2}\right) = 2\cdot\frac{1}{8} - 3\cdot\frac{1}{4} - 6 + 5$$

$$= \frac{1}{4} - \frac{3}{4} - 1 = -1\frac{1}{2}$$

#2 $f(x)=\dfrac{x^2+1}{x^2-4}$

$D: \{x \mid x \neq \pm 2\}$



$$f'(x)=\frac{-10x}{(x^2-4)^2}$$

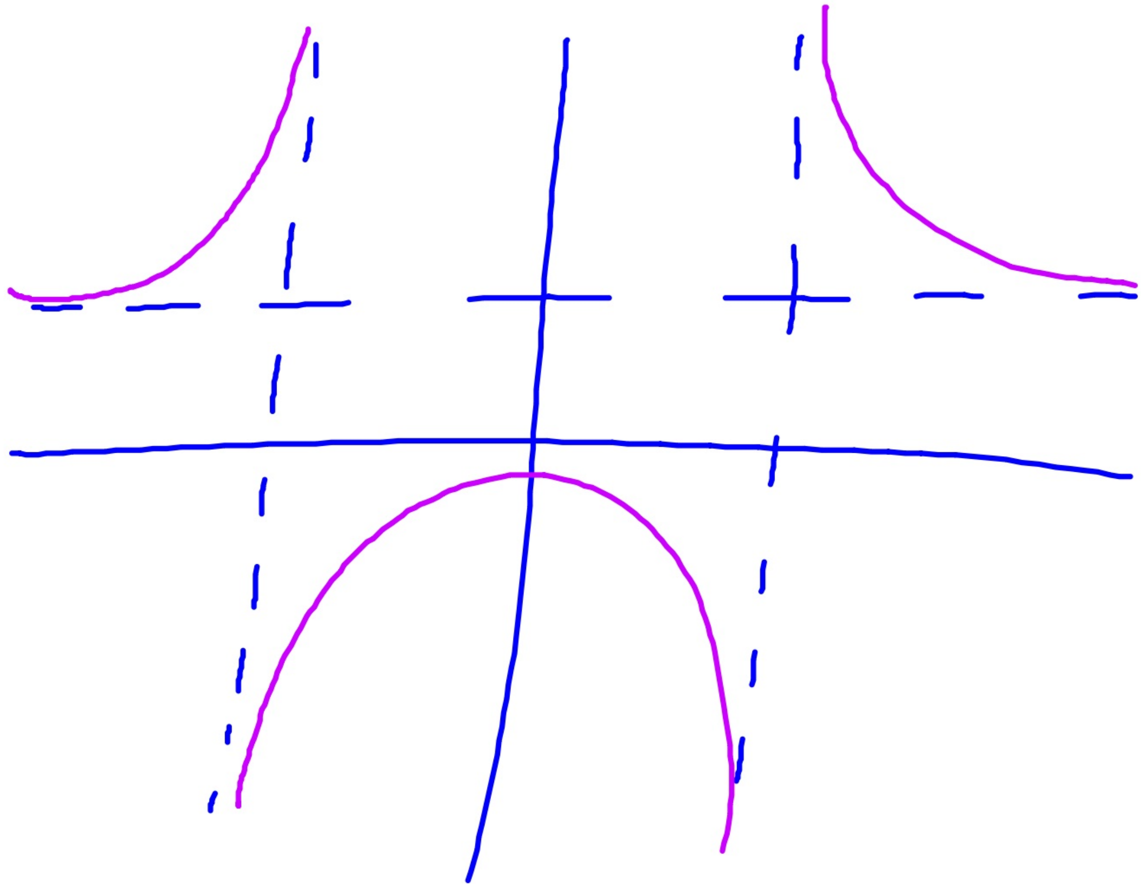
$$=-10x(x^2-4)^{-2}$$

$$=-10\left(x(x^2-4)^{-2}\right)$$

$$f''(x)=-10\left(\underline{x\cdot -2(x^2-4)^{-3}(2x)}+\underline{(x^2-4)^{-2}\cdot 1}\right)$$

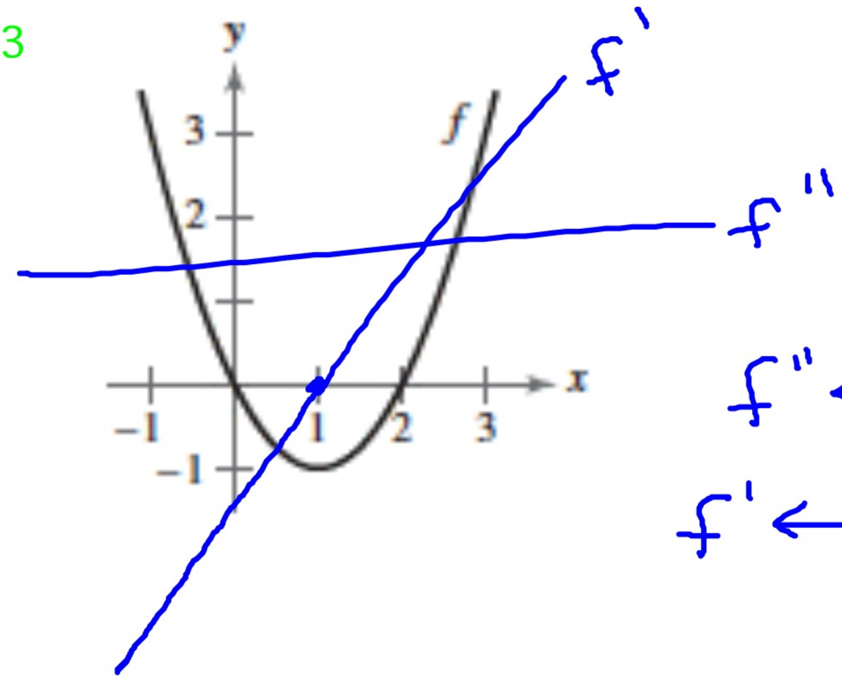
$$=-10\left((x^2-4)^{-3}\left(-4x^2+(x^2-4)^1\right)\right)$$

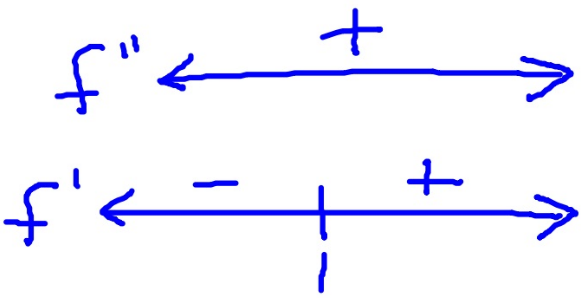
$$=\frac{-10(-3x^2-4)}{(x^2-4)^3}=\frac{30x^2+40}{(x^2-4)^3}$$



Sketch f' and f''.

#3





$$y = ax^2 + bx + c$$

$$y' = 2ax + b$$

$$0 = 2ax + b$$

$$\frac{-b}{2a} = x$$

We use the first derivative to find:

intervals of increasing/decreasing and

relative extrema

We use the second derivative to find:

intervals of concavity and

points of inflection

But, there are two tests that have an official name

First Derivative Test

Second Derivative Test

BOTH tests are ways to locate relative extrema

local

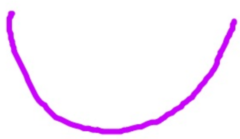
THEOREM 3.9 SECOND DERIVATIVE TEST

Let f be a function such that $\boxed{f'(c) = 0}$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

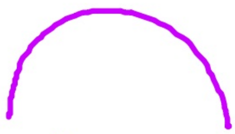
If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

$f''(c) > 0$



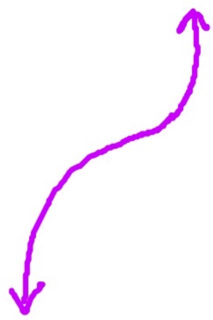
rel. min

$f''(c) < 0$



rel. max

$f''(c) = 0$



$f(4) = 7$

$f'(4) = 0$

$f''(4) = -8$

$x = 4$

rel. max

$\cap$

$f'(-3) = 0$

$f''(-3) = 12$

$x = -3$

rel. min

$\cup$

Find the relative extrema using the second derivative test.

#1 $f(x) = -3x^5 + 5x^3$

$f'(x) = -15x^4 + 15x^2$

$0 = -15x^2(x^2 - 1)$

$x = 0, 1, -1$

$f''(x) = -60x^3 + 30x$

$f''(x) = -30x(2x^2 - 1)$

$f''(0) = 0$ not rel. or rel. min

$f''(1) < 0$ rel. max @ $(1, 2)$

$f''(-1) > 0$ rel. min @ $(-1, -2)$

#2 $f(x) = x^3 - 5x^2 + 7x$

increasing/decreasing intervals
relative extrema (1st derivative test)
relative exrema (2nd derivative test)
concavity
POI
Sketching f' and f''
Surprise from last quiz