2.3 Product and Quotient Rules and Higher-Order Derivatives

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.
- Find a higher-order derivative of a function.

The Product Rule

$$\frac{d}{dx}[fg] = f \cdot g' + g \cdot f'$$

#1

$fg' + gf'$

$f(x) = (6x + 5)(x^3 - 2)$

$f(x) = 6x^4 - 12x + 5x^3 - 10$

$f'(x) = 24x^3 - 12 + 15x^2$

$f(x) = \underbrace{(6x+5)}_{f}\underbrace{(x^3-2)}_{g}$

$f'(x) = (6x+5)3x^2 + (x^3-2)6$

$= 18x^3 + 15x^2 + 6x^3 - 12$

$= 24x^3 + 15x^2 - 12$

#2: Write the equation of the line tangent to f(x) at x = 0

$(0, 0)$

$$f(x) = \underset{f}{\sin x}\,\underset{g}{\cos x}$$

$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$

$f'(x) = -\sin^2 x + \cos^2 x$

$f'(0) = 1$

$y - 0 = 1(x - 0)$

$\boxed{y = x}$

Find f'(x) and f'(c)

#3 $f(x) = \overset{f}{x}\,\overset{g}{\cos x}$ $\qquad c = \frac{\pi}{4}$

$f'(x) = x(-\sin x) + \cos x \cdot 1$

$f'(x) = -x\sin x + \cos x$

$f'\left(\frac{\pi}{4}\right) = \frac{-\pi}{4} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{-\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2}$

#4 $f(x) = \frac{\sin x}{x} = \overset{f}{x^{-1}} \cdot \overset{g}{\sin x}$ $\qquad c = \frac{\pi}{6}$

$f'(x) = x^{-1} \cdot \cos x + \sin x \cdot (-1x^{-2})$

$= \frac{\cos x}{x} - \frac{\sin x}{x^2}$

$f'\left(\frac{\pi}{6}\right) = \frac{\frac{\sqrt{3}}{2}}{\frac{\pi}{6}} - \frac{\frac{1}{2}}{\pi^2/36} = \frac{3\sqrt{3}}{\pi} - \frac{18}{\pi^2}$

Instead of re-writing, you can use the Quotient Rule

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$$

$$\frac{d}{dx}\left[\frac{Hi}{Lo}\right] = \frac{Lo \cdot dHi - Hi \cdot dLo}{Lo^2}$$

$$y = \frac{\sin x}{x} \qquad y' = \frac{\cos x}{x} - \frac{\sin x}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

$$y = \frac{\sin x}{x} \qquad y' = \frac{x \cdot \cos x - \sin x \cdot 1}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

#5

$$y = \frac{\sqrt{x}}{2x-1}$$

$$y' = \frac{(2x-1)\frac{1}{2}x^{-1/2} - x^{1/2}\cdot 2}{(2x-1)^2}$$

$$y' = \frac{x^{1/2} - \frac{1}{2}x^{-1/2} - 2x^{1/2}}{(2x-1)^2}$$

$$y' = \frac{-\frac{2}{2}x^{1/2} - \frac{1}{2}x^{-1/2}}{(2x-1)^2} = \frac{-\frac{1}{2}x^{-1/2}(2x^1 + 1)}{(2x-1)^2} = \frac{-(2x+1)}{2\sqrt{x}(2x-1)^2}$$

$x^{1/2}$

$x^7 - x^5$

$x^5(x^2-1)$

#6 Determine the point(s) at which the graph of f(x) has a horizontal tangent line.

f(x) = (x - 4)/(x^2 - 7)

$$f(x) = \frac{x-4}{x^2-7}$$

$$f'(x) = \frac{(x^2-7)\cdot 1 - (x-4)2x}{(x^2-7)^2}$$

$$= \frac{x^2-7-2x^2+8x}{(x^2-7)^2}$$

$$= \frac{-x^2+8x-7}{(x^2-7)^2}$$

$$= \frac{-(x^2-8x+7)}{(x^2-7)^2} = \frac{-(x-7)(x-1)}{(x^2-7)^2}$$

$$0 = \frac{-(x-7)(x-1)}{(x^2-7)^2}$$

$$\boxed{\left(1, \tfrac{1}{2}\right); \left(7, \tfrac{1}{14}\right)}$$

for horizontal tangents, look at the numerator of a rational function.

#7

Find the derivative of f(x) = tanx

$$f(x) = \frac{\sin x}{\cos x} \qquad f'(x) = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$f'(x) = \sec^2 x$$

#8

Find the derivative of y = cotx

$$y = \frac{\cos x}{\sin x} \qquad y' = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x$$

#9

Find the derivative of y = secx

$$y = \frac{1}{\cos x}$$

$$y' = \frac{\cos x\,(0) - (1)(-\sin x)}{\cos^2 x}$$

$$y' = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \tan x \sec x$$

THEOREM 2.9 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

#10

$$y = \frac{1 - \cos x}{\sin x}$$

#11: Find the equation of the tangent line to the point.

$$h(t) = \frac{\sec t}{t} \qquad \left(\pi, -\frac{1}{\pi}\right)$$

$$h'(t) = \frac{t \sec t \tan t - \sec t \cdot 1}{t^2}$$

$$h'(\pi) = \frac{0 - (-1)}{\pi^2}$$

$$= \frac{1}{\pi^2}$$

$$y + \frac{1}{\pi} = \frac{1}{\pi^2}(x - \pi)$$

Are these functions ''quotient rule necessary''?

(or ''product rule necessary''?)

A $y = \dfrac{3x^2 + 7x}{x}$

D $y = \dfrac{4x^{3/2}}{x}$

B $f(x) = \dfrac{x}{x^2 + 1}$

C $g(s) = \sqrt{s}(s^2 + 8)$

E $y = \dfrac{6}{7x^2}$

F $y = (x-3)^2$

Higher Order Derivatives

$$s(t) \qquad \text{Position function}$$
$$v(t) = s'(t) \qquad \text{Velocity function}$$
$$a(t) = v'(t) = s''(t) \qquad \text{Acceleration function}$$

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	rate of change of position
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	rate of change of velocity
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	rate of change of acceleration
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	
$\vdots$				
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	

$$y = 7x^5 - 3x^4 + 11x^2 - 3x + 4$$

$$y' = 35x^4 - 12x^3 + 22x - 3$$

$$y'' = 140x^3 - 36x^2 + 22$$

$$y''' = 420x^2 - 72x \qquad y^{(6)} = 0$$

$$y^{(4)} = 840x - 72$$

$$y^{(5)} = 840$$

First Derivative: $\frac{dy}{dx}$

Second Derivative: $\frac{d^2y}{dx^2}$

Third Derivative:

$$\frac{d^3y}{dx^3}$$

*#12: Find f''(x)

$1+\tan^2 x = \sec^2 x$

$f(x) = \sec x$

$f'(x) = \tan x \sec x$

$f''(x) = \tan x \sec x \tan x + \sec x \sec^2 x$

$= \sec x(\tan^2 x + \sec^2 x)$

$= \sec x(\sec^2 x - 1 + \sec^2 x)$

$= \sec x(2\sec^2 x - 1)$