## AB Calculus

## MVT and other Theorems

1. Determine whether the statements must be true, might be true or cannot be true.
a. If $f(1)<0$ and $f(5)>0$, then there must be a number $c$ in $(1,5)$ such that $f(c)=0$
b. If $f$ is continuous on $[1,5]$ and $f(1)<0$ and $f(5)>0$, then there must be a number $c$ in $(1,5)$ such that $f(c)=0$.
c. If $f$ is continuous on $[1,5]$ and $f(1)=2$ and $f(5)=2$, then there must be a number $c$ in $(1,5)$ such that $f^{\prime}(c)=0$
d. If $f$ is differentiable on $[1,5]$ and $f(1)=2$ and $f(5)=2$, then there must be a number $c$ in $(1,5)$ such that $f^{\prime}(c)=0$.
e. If the function has $3 x$-intercepts, then $f$ must have at least 2 points at which its tangent line is horizontal.
2. The functions f and g are twice differentiable for all real numbers. The table below shows values of the functions and their derivatives at selected values of x . The function h is given by $h(x)=f(x) g(x)$.

| $\boldsymbol{x}$ | $\boldsymbol{f}(\boldsymbol{x})$ | $\boldsymbol{g}(\boldsymbol{x})$ | $\boldsymbol{f}^{\prime}(\boldsymbol{x})$ | $\boldsymbol{g} \boldsymbol{g}(\boldsymbol{x})$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 6 | 4 | -4 | 3 |
| 2 | 9 | -1 | -1 | 1 |
| 3 | 5 | 1 | 0 | -2 |
| 4 | -3 | 3 | 5 | 6 |

a. Estimate $h^{\prime}(2.4)$.
b. Explain why there must be a value $r$ for $1<r<4$ such that $h(r)=-2$.
c. Explain why there must be a value p for $1<\mathrm{p}<4$ such that $\mathrm{h}^{\prime}(\mathrm{p})=-11$.
3. Let $f$ be a twice-differentiable function such that $f(2)=5$ and $f(5)=2$. Let $g$ be the function given by $g(x)=f(f(x)$ and $h$ be the function given by $h(x)=f(x)-x$.
a. Explain why there must be a value c in $2<\mathrm{c}<5$ such that $\mathrm{f}^{\prime}(\mathrm{c})=-1$
b. Show $\mathrm{g}^{\prime}(2)=\mathrm{g}^{\prime}(5)$. Use this result to explain why there must be a value k on $2<\mathrm{k}<5$ such that $g^{\prime \prime}(k)=0$.
c. Explain why there must be a value $r$ for $2<r<5$ such that $h(r)=0$.
4. The continuous function $f$ (given below) is defined on the interval $[-4,3]$. The graph consists of two quarter circles and one line segment.


Find the average rate of change on the interval $[-4,3]$. There is no point $c$, on the interval $(-4,3)$ for which $f^{\prime}(c)$ is equal to the average rate of change. Explain why this statement does not contradict the Mean Value Theorem.
5. A particle moves along a horizontal line with a positive velocity, $v(t)$, where $v$ is a differentiable function of $t$. The time t is measured in seconds and the velocity is measured in $\mathrm{cm} / \mathrm{sec}$. The velocity of the particle at selected times is given in the table below.

| $\mathrm{t}(\mathrm{sec})$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{v}(\mathrm{t}) \mathrm{cm} / \mathrm{sec}$ | 37 | 17 | 5 | 1 | 6 | 17 | 38 |

a. What is the smallest number of times at which the velocity of the particle could equal $20 \mathrm{~cm} / \mathrm{sec}$ in the open interval $(0,12)$ seconds? Justify your answer.
b. Must there be a time $c$ in the interval $(0,12)$ such that $v^{\prime}(c)=0$ ? Justify your answer.
c. Find the average acceleration of the particle over the time interval $(8,10)$ seconds. Include units of measure.

## 6.

1998 \#91 (AB but suitable for BC) - Calc OK:Let $f$ be a function that is differentiable on the open interval ( 1,10 ). If $f(2)=-5, f(5)=5$, and $f(9)=-5$, which of the following must be true?
I. $f$ has at least 2 zeros.
II. The graph of $f$ has at least one horizontal tangent.
III. For some $c, 2<c<5, f(c)=3$.
a. None
c. I and II only
b. I only
d. I and III only
e. I, II and III
7.

2003 \#83 (BC) - Calc OK:

| $x$ | 0 | 1 | 2 | 3 | 4 |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $f(x)$ | 2 | 3 | 4 | 3 | 2 |

The function $f$ is continuous and differentiable on the closed interval [0,4]. The table above gives selected values of $f$ on this interval. Which of the following statements must be true?
a. The minimum value of $f$ on $[0,4]$ is 2 .
b. The maximum value of $f$ on $[0,4]$ is 4 .
c. $f(x)>0$ for $0<x<4$
d. $f^{\prime}(x)<0$ for $2<x<4$
e. There exists $c$, with $0<c<4$, for which $f^{\prime}(c)=0$.

## 8.

1997 \#81 (BC) - Calc OK: Let $f$ be a continuous function on the closed interval [-3.6]. If $f(-3)=-1$ and $f(6)=3$, then the Intermediate Value Theorem guarantees that
a. $\quad f(0)=0$
b. $\quad f^{\prime}(c)=\frac{4}{9}$ for at least one $c$ between -3 and 6.
c. $-1 \leq f(x) \leq 3$ for all $x$ between -3 and 6 .
d. $f(c)=1$ for at least one $c$ between -3 and 6 .
e. $f(c)=0$ for at least one $c$ between -1 and 3.
1)
a. Might be true, but if the function is not continuous it could be false.
b. The IVT guarantees this is always true.
c. Might be true, but if the function is not differentiable it could be false.
d. MVT or Rolle's Theorem guarantee this is always true.
e. Might be true, but if the function is not differentiable it could be false.
2)
a. $h^{\prime}(2.4) \approx 14$
b. and c.

Since $f$ and $g$ are continuous functions (because they are differentiable), so is $h(x)=f(x) \cdot g(x) \Rightarrow$ Intermediate Value Theorem applies to $h(x)$ in [1,4]. With $\left.\begin{array}{l}h(1)=f(1) \cdot g(1)=6 \cdot 4=24 \\ h(4)=f(4) \cdot g(4)=(-3) \cdot 3=-9\end{array}\right\} \Rightarrow$ there must be a value $r$ for
$1<r<4$ such that $-9=h(4)<h(r)=-2<h(1)=24$.
Since $f$ and $g$ are continuous and differentiable functions, so is $h(x)=f(x) \cdot g(x) \Rightarrow$ Mean Value Theorem applies to $h(x)$ in $[1,4]$. With $h(1)=f(1) \cdot g(1)=6 \cdot 4=24$
$h(4)=f(4) \cdot g(4)=(-3) \cdot 3=-9\} \Rightarrow \frac{h(4)-h(1)}{4-1}=-11$
So there must be a value $p$ for $1<p<4$ such that $h^{\prime}(p)=\frac{h(4)-h(1)}{4-1}=-11$.
3)
(a) The Mean Value Theorem guarantees that there is a value $c$, with $2<c<5$, so that
$f^{\prime}(c)=\frac{f(5)-f(2)}{5-2}=\frac{2-5}{5-2}=-1$.
(b) $g^{\prime}(x)=f^{\prime}(f(x)) \cdot f^{\prime}(x)$
$g^{\prime}(2)=f^{\prime}(f(2)) \cdot f^{\prime}(2)=f^{\prime}(5) \cdot f^{\prime}(2)$
$g^{\prime}(5)=f^{\prime}(f(5)) \cdot f^{\prime}(5)=f^{\prime}(2) \cdot f^{\prime}(5)$
Thus, $g^{\prime}(2)=g^{\prime}(5)$.
Since $f$ is twice-differentiable, $g^{\prime}$ is differentiable everywhere, so the Mean Value Theorem applied to $g^{\prime}$ on [ 2,5 ] guarantees there is a value $k$, with $2<k<5$, such that $g^{\prime \prime}(k)=\frac{g^{\prime}(5)-g^{\prime}(2)}{5-2}=0$.
(c)

Let $h(x)=f(x)-x$.
$h(2)=f(2)-2=5-2=3$
$h(5)=f(5)-5=2-5=-3$
Since $h(2)>0>h(5)$, the Intermediate Value Theorem guarantees that there is a value $r$, with $2<r<5$, such that $h(r)=0$.

The average rate of change of $f$ on the interval $-4 \leq x \leq 3$ is
$\frac{f(3)-f(-4)}{3-(-4)}=-\frac{2}{7}$.
To apply the Mean Value Theorem, $f$ must be differentiable at each point in the interval $-4<x<3$. However, $f$ is not differentiable at $x=-3$ and $x=0$.
5)
a. Two times. Since $v(0)=37>20>17=v(2)$ and $v(10)=17<20<38=v(12)$ and $v$ is continuous, by the Intermediate Value Theorem, there exists a value ${ }^{t_{1}}$ in $(0,2)$ such that $v\left(t_{1}\right)=20$ and a value $t_{2}$ in $(10,12)$ such that $v\left(t_{2}\right)=20$. (We know that $v$ is differentiable on its domain so it is also continuous there.)
b. One time. Since $v(2)=v(10)=17$ and $v$ is continuous and differentiable, by the Mean Value Theorem, there exists a value $t$ in $(2,10)$ such that:

$$
a(t)=v^{\prime}(t)=\frac{17-17}{10-2}=0
$$

Or, since $v\left(t_{1}\right)=v\left(t_{2}\right)=20$ and $t_{1}<t_{2}$ for the $t_{1}$ and $t_{2}$ found in part (a), there exists a value $t$ in $\left(t_{1}, t_{2}\right)$ such that:

$$
a(t)=v^{\prime}(t)=\frac{20-20}{t_{2}-t_{1}}=0
$$

c. Average acceleration $=$

$$
\frac{v(10)-v(8)}{10-8}=\frac{17-6}{2}=5.5 \frac{\mathrm{~cm} / \mathrm{sec}}{\mathrm{sec}}=\frac{\mathrm{cm}}{\mathrm{sec}^{2}}
$$

6) e
7) e
8) $d$
