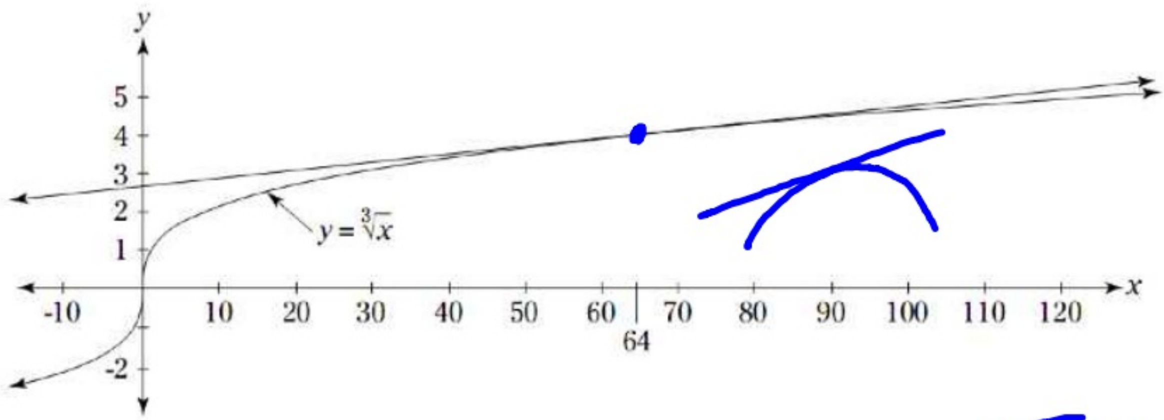
§3.7—Linearization & Differentials

Linear approximation is a very easy thing to do, and once you master it, you can impress all of your friends by calculating things like $\sqrt[3]{70}$ in your head . . . . about 4.125! Impressed? I'll teach you how.

Recall that if a function $f(x)$ is differentiable at $x=c$, we say it is locally linear at $x=c$. This means that as we zoom in closer and closer and closer and closer around $x=c$, the graph of $f(x)$, regardless of how curvy it is, will begin to look more and more and more and more like the tangent line at $x=c$.

This means that we can use the equation of the tangent line of $f(x)$ at $x=c$ to approximate $f(c)$ for values close to $x=c$. Let's take a look at $\sqrt[3]{70}$ and the figure below.



$\sqrt[3]{70} \approx 4.1212$

Example 1:
Approximate $\sqrt[3]{70}$ by using a tangent line approximation centered at $x=64$. Determine if this approximation is an over or under-approximation. Approximate $\sqrt[3]{70}$ using a secant line approximation using $x=64$ and ~~$x=125$~~. Determine if this approximation is an over or under-approximation.

$y = \sqrt[3]{x}$

$y' = \frac{1}{3}x^{-2/3}$

$y'(64) = \frac{1}{48}$

$y'' = \frac{-2}{9}x^{-5/3}$

$y''(64) < 0$; concave down

$\therefore$ over-approx.

$(64, 4)$

$y - 4 = \frac{1}{48}(x - 64)$

$y - 4 = \frac{1}{48}(70 - 64)$

$y = 4\frac{1}{8} = 4.125$

How to find linear approximations of $f(x)$ at $x = c$, the center to approximate $f(x)$ at $x = a$, a value near the center $x = c$.

1. Find the equation of the tangent line at the center $(c, f(c))$ in point-slope form.
2. Solve for y and rename it $L(x)$.
3. Plug in $x = a$ into $L(x)$ writing the notation VERY CAREFULLY as $f(a) \approx L(a) =$
4. If asked, determine if $L(a)$ is an over-approximation or an under approximation by examining the concavity of $f(x)$ at the center $x = c$.
    a. If $f''(c) < 0$, $f(x)$ is concave down at $x = c$ then $L(a)$ is an over-approximation
    b. If $f''(c) > 0$, $f(x)$ is concave up at $x = c$ and $L(a)$ is an under-approximation

Example 2:
Estimate the fourth root of 17. Determine if the linearization is and over- or under-approximation.

$y = x^{1/4}$

$y' = \frac{1}{4}x^{-3/4}$

$y'(16) = \frac{1}{32}$

$y''(16) < 0 \therefore$ CCD; overapprox.

$y - 2 = \frac{1}{32}(x - 16)$

$y - 2 = \frac{1}{32}(17 - 16)$

$y = 2\frac{1}{32}$

Example 3:
Approximate 3.01^5. Determine if the linearization is and over- or under-approximation.

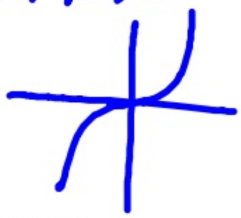
$y = x^5$ $\quad (3, 243)$ $\quad y - 243 = 405(x - 3)$

$y' = 5x^4$

$y'(3) = 405$

$y = 247.05$

$y''(3) > 0$; CCU (under approx)



Example 4:
Approximate $\ln\left(e^{10} + 5\right)$. Determine if the linearization is and over- or under-approximation.