3.6—Optimization

Optimization is a useful application of differential calculus. Any time you use the superlative case—biggest, smallest, cheapest, strongest, ugliest, etc.—you're trying to optimize. What's keeping us from making something infinitely big, infinitely small, and infinitely ugly? Well, there is always some limiting factor, a constraint that prevents that from happening.

In this unit, we will be looking for **Absolute Extrema**, but we won't always have endpoints. We'll learn 3 different methods for justifying that the value we find is indeed the value that yields a Global Extrema. Each problem is unique, and we'll have to decide which justification method to use so that we expend the LEAST amount of effort.

These types of problems are among the most challenging for Calculus students because they combine Math with English, they're the much dreaded WORD PROBLEMS. Careful reading can go a long way in translating the written language into the language of mathematics, and a picture, they say, is worth 10^3 words.

Here's what the Greek youth used to do for recreation before it was cool to use (Twitter/Facebook/Instagram . . .)

Example 1:
Find two positive numbers whose sum is 60 and whose product of one times the square of the other is a maximum. Don't get your toga in a knot!!

Primary Equation: Maximize

$P = xy^2$

$P(y) = (60-y)y^2$

$P(y) = 60y^2 - y^3$

$P'(y) = 120y - 3y^2$

$0 = 3y(40-y)$

$y = 0, 40$

P': (0) $+$ 40 $-$

Secondary Equation

$x + y = 60$

$x = 60 - y$

40

$\boxed{40, 20}$

Example 2:
An 8-inch long pipe cleaner (used to clean pipes and form rectangles for a calculus class example) is bent into the shape of a rectangle. What dimensions of the rectangle will produce the rectangle with maximum area?

Calculus Ninja (Another useful application of a pipe cleaner)

Primary Equation

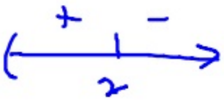
$A = xy$

$A(x) = x(4-x)$

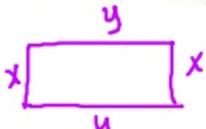
$A(x) = 4x - x^2$

$A'(x) = 4 - 2x$

$2 = x$



Secondary



$2x + 2y = 8$

$x + y = 4$

$y = 4 - x$

$\boxed{2\text{in}, 2\text{in}}$

Example 3:
A ladybug farmer has 500 inches of fencing and wants to fence off a rectangular field that borders on a straight river (to enclose his grazing ladybugs). He needs no fence along the river (ladybugs can't swim, and he has clipped their wings). What are the dimensions of the field that has the largest grazing area for his hungry ladybugs?

Primary

$A = xy$

$A(x) = x(500 - 2x)$

$A(x) = 500x - 2x^2$

$A'(x) = 500 - 4x$

$125 = x$

$A'(125) = 0$

$A''(125) = -4$

Since only 1 crit. #, $x = 125$ is an abs. max.

Secondary

$2x + y = 500$

$y = 500 - 2x$

$\boxed{125\text{in}, 250\text{in}}$

Example 4:
The same ladybug farmer has purchase some expensive, extraordinary diva ladybugs who require ***exactly*** 10,000 square inches of grazing in order to be at their optimal "ladybug-like" state. What is the least amount of fencing required to get these diva ladybugs at their optimal state if the farmer is still allowed to build along the very straight river?? (Don't think diva lady bugs can swim and don't still have their wings clipped).

Primary Equation

$P = 2x + y$

$P = 2x + \frac{10{,}000}{x}$

$P'(x) = 2 - \frac{10{,}000}{x^2}$

$x = \sqrt{5000}$

Secondary Equation

$xy = 10{,}000$

$y = \frac{10{,}000}{x}$

$y = \frac{10{,}000}{\sqrt{5000}}$

$P = 2\sqrt{5000} + \frac{10{,}000}{50\sqrt{2}} = 100\sqrt{2} + \frac{200}{\sqrt{2}}$

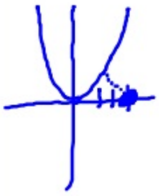
$= 200\sqrt{2}$ in

Sometimes, endpoints are relevant.

Example 5:

Find the point on the curve $y = x^2$ closest to $(3,0)$.

Secondary



minimizing distance

primary: $d = \sqrt{(x-3)^2 + (y-0)^2}$

$d = \sqrt{(x-3)^2 + x^4}$

$D = (x-3)^2 + x^4$

$D' = 2(x-3) + 4x^3$

$0 = 4x^3 + 2x - 6$

$0 = 2(2x^3 + x - 3)$

$x = 1$ $-$ | $+$ at 1

$\boxed{(1,1)}$

Because d is the smallest when the expression inside the radical is smallest, you need only find the critical numbers of the radicand when maximizing or minimizing distance.

WAYS TO JUSTIFY AN ABSOLUTE EXTREMA When you have relevant *ENDPOINTS*

Method 1: Closed interval argument (EVT) **When you have relevant *ENDPOINTS***

We can play the "biggest y-value" game with Team Endpoint and Team Critical Value. This sure-fire method requires finite endpoints and a continuous function over the relevant interval. This method will not work every time, because some examples may have at least one infinite (unbounded) endpoint. Also, once we reduce the primary equation down to a single variable, many equations will no longer be continuous. Alas, the method fails.

This works **when you have relevant *ENDPOINTS***

Example 6:

A manufacturer wants to design an open box having a square base with a surface area of 108 square inches. What dimensions will produce a maximum volume?

Primary

$V = x^2 y$

$V = x^2\left(\frac{108 - x^2}{4x}\right)$

$V = \frac{1}{4}x(108 - x^2)$

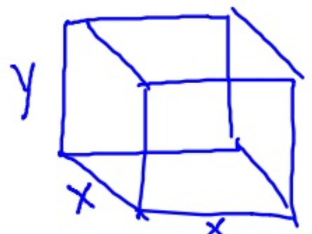
$V'(x) = 27 - \frac{3}{4}x^2$

$6 = x$ $+$ | $-$ on 0 to 6

Secondary

$108 = x^2 + 4xy$

$\frac{108 - x^2}{4x} = y$



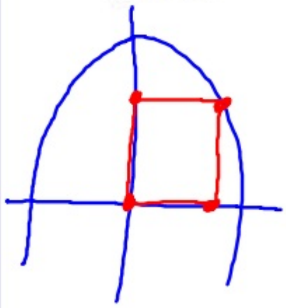
5 sides

$\boxed{6\text{in}, 6\text{in}, 3\text{in}}$

Example 7:

Secondary

Find the dimensions of the largest rectangle that can be inscribed under the curve $y = 16 - x^2$ in the first quadrant.



Primary

$A = xy$

$A(x) = x(16 - x^2)$

$A'(x) = 16 - 3x^2$

$\frac{4}{\sqrt{3}} = x$

$\boxed{\frac{4}{\sqrt{3}}, \frac{32}{3}}$

<u>Steps to a successful Optimization Problem:</u>

1. Read the problem carefully
2. Draw a picture. Label unknowns as variables. Label constants as numbers. IT IS EASIER TO LABEL SMALL PARTS AS SINGLE VARIABLES SO THAT YOU CAN AVOID FRACTIONS AND/OR SUBTRACTING.
3. Write a primary equation for the quantity to be optimized.
4. Identify the limiting factor/constraint and write a secondary equation (sometimes) involving this constraint.
5. Solve the secondary equation for any convenient variable and plug it into the primary equation to establish an equation for the optimal quantity in terms of a single variable.
6. Simplify the primary equation and think about a **relevant/feasible domain**.
7. Differentiate and find critical values within the relevant domain.
8. Determine the Absolute Extrema and JUSTIFY.
9. Answer the question in a complete sentence using appropriate units.
10. Smile ☺

WAYS TO JUSTIFY AN ABSOLUTE EXTREMA (in absence of endpoints)

<u>Method 2</u>: 1st Derivative Test for Relative Extrema modified for Absolute Extrema
We will use this method when we don't have both endpoints at a closed interval, but instead have either a half-open interval or open interval. It does require a continuous function, though.

<u>Method 3</u>: 2nd Derivative Test for Relative Extrema modified for Absolute Extrema
This method is preferred to method 2 when the 2nd derivative is easy to obtain.

**In either case, it is important to show the test works FOR ALL values in the relevant domain. This transforms each test from a local argument to a global argument.

Example 8:
An open-top box is to be constructed from a square piece of cardboard, 3 feet wide, by cutting out a square from each corner and bending up the sides. What is the largest volume that the box can have?

Primary

$V = x^2 y$

$V(x) = x^2\left(\frac{3-x}{2}\right)$

$V(x) = \frac{3}{2}x^2 - \frac{1}{2}x^3$

$V'(x) = 3x - \frac{3}{2}x^2$

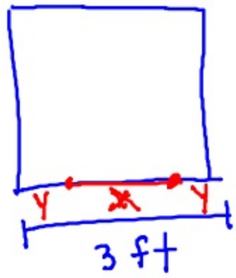
$0 = 3x - \frac{3}{2}x^2$

$2 = x$

$y = \frac{1}{2}$

$V = 2^2 \cdot \frac{1}{2}$

$= 2 \text{ in}^3$



$x + 2y = 3$

secondary

Example 9:
A cylindrical aluminum soda can is to hold 12 ounces of tasty beverage. What are the dimensions of the can that will minimize the amount of aluminum used? Hint: 12 US fluid ounces equals 21.6562 cubic inches. If the dimensions of the actual can are 2.6 inch diameter and 4.75 inches tall, does the actual can have the optimal dimensions?

Sometimes we need to establish a pattern to write our primary equation.

Example 10:
The manager of an 80-unit apartment complex is trying to decide what rent to charge. Experience has shown that at a rent of \$200, all of the units will be full. On the average, one additional unit will remain vacant for each \$20 increase in rent. Find the rent to charge so as to maximize revenue.

It's fun to inscribe shapes into other shapes that circumscribe the inscribed shape we're inscribing.

Example 11:
Find the volume of the largest cone that can be inscribed inside a sphere of radius 5.

Example 12:

An arch top window is being built whose bottom is a rectangle and the top is a semicircle. If there is 12 meters of framing materials what is the width of the window that lets in the most light?

Example 13:

Find an equation (in general form) for the tangent line to the graph of $f(x)=\frac{2}{3}-2x-x^2-\frac{1}{3}x^3$ having maximum slope.