2.6 Derivatives of Inverse Functions

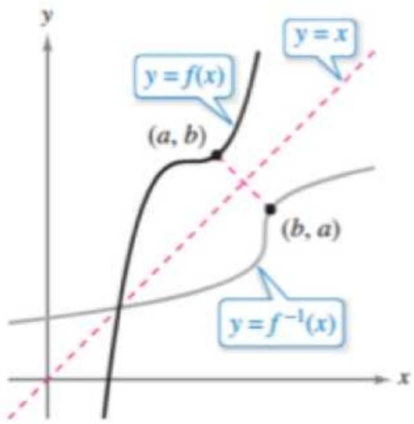
Review: Two functions are inverse functions if

$f(f^{-1}(x))$

$$\boxed{\begin{gathered}\left(f \circ f^{-1}\right)(x) = x \\ \text{and} \\ \left(f^{-1} \circ f\right)(x) = x\end{gathered}}$$

Inverse Properties

- The graphs of f and f^{-1} are reflections about the line y=x.
- The domain of f is the range of f^{-1}.
- The range of f is the domain of f^{-1}.
- A function has an inverse if and only if it is one-to-one.



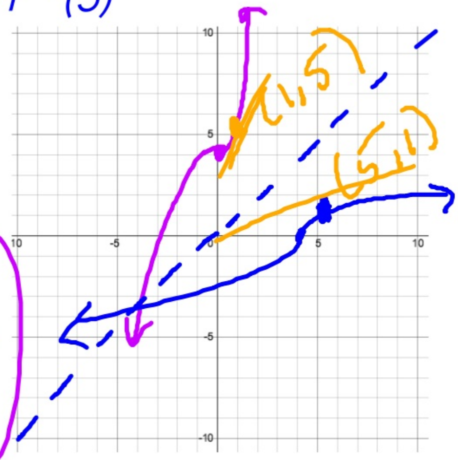
Sketch. Compare the slopes at f(1) and $f^{-1}(5)$

$f(x) = x^3 + 4$ $\qquad$ $f^{-1}(x) = \sqrt[3]{x-4}$

$f'(x) = 3x^2$ $\qquad$ $(f^{-1})'(x) = \frac{1}{3}(x-4)^{-2/3}$

$f'(1) = 3$ $\qquad$ $(f^{-1})'(5) = \frac{1}{3}$

reciprocal



1) $f(x) = 2x^3 + 3x$

Verify the function is 1:1 (monotonic: a function that is always increasing or always decreasing)

$f'(x) = 6x^2 + 3 > 0$

Find the derivative of the inverse at x = 5. $\boxed{(f^{-1})'(5) = \frac{1}{9}}$

$5 = 2x^3 + 3x$

$1 = x$

$f'(x) = 6x^2 + 3$

$f'(1) = 9$

reciprocal

$f: (1, 5)$

$f^{-1}: (5, 1)$

2) $f(x) = x^3 - \frac{4}{x}$ $\quad$ $Domain: (0, \infty)$

f(x) and g(x) are inverses. Find $g'(6) = \frac{1}{13}$

$6 = x^3 - \frac{4}{x}$

$2 = x$

$f'(x) = 3x^2 + 4x^{-2}$

$f'(2) = 13$

reciprocal

$f: (2, 6)$

$g: (6, 2)$

ex: If $f(x) = x^5 + 2x^3 + x - 1$ and $(f \circ g)(x) = (g \circ f) = x$ find $g'(3)$. $\underline{\quad \frac{1}{12} \quad}$

$$3 = x^5 + 2x^3 + x - 1$$

$$1 = x$$

$$f'(x) = 5x^4 + 6x^2 + 1$$

$$f'(1) = 12$$

$f : (1, 3)$

$g : (3, 1)$

3. 18% answered correctly

If $f(x) = x^3 + x$ and $h(x)$ is the inverse of $f(x)$, then $h'(2)$ is

A) $\frac{1}{13}$ B) $\frac{1}{4}$ C) 1 D) 4 E) 13

$2 = x^3 + x$

$1 = x$

$f: (1, 2)$

$h: (2, 1)$

$f'(x) = 3x^2 + 1$

$f'(1) = 4$

14% of students answered this correctly

4) Let f be a differentiable function with f(3) = 15, f'(3) = -8, f'(6) = -2, f(6) = 3. The function g is differentiable and $g(x) = f^{-1}(x)$. What is the value of g'(3)?

a) -1/8

b) -1/2

c) 1/6

d) 1/3

e) not possible

f : (6, 3)

g : (3, 6)

$f'(6) = -2$

g(x) (3, 6)

$m = -\frac{1}{2}$

$y - 6 = -\frac{1}{2}(x - 3)$

5.

Suppose f is a one-to-one function, which is differentiable for all real numbers x. The table below gives some of the values of $f(x)$ and $f'(x)$:

x	$f(x)$	$f'(x)$
1	2	$\frac{7}{6}$
2	3	$\frac{7}{6}$
3	5	$\frac{19}{6}$
4	10	$\frac{43}{6}$

(a) Write an equation of the tangent line, T_1, to the function $f(x)$ at $x = 3$.

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(b) Write an equation of the normal line, N_1, to the function $f(x)$ at $x = 3$.

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(c) Write an equation of the tangent line, T_2, to the function $f^{-1}(x)$ at $x = 3$.

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Mean Score 0.95

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

$g'(1) = 5$

$y - 1 = \frac{1}{5}(x-2)$

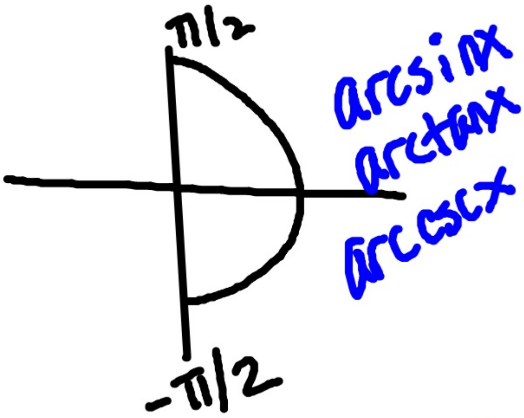
$g\ (1, 2)$

$g^{-1}\ (2, 1)$

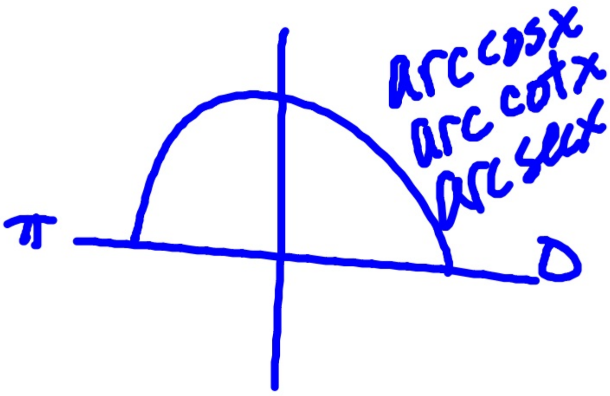
2.6 Derivatives of Inverse Functions Cont.

ex: Evaluate.

a) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

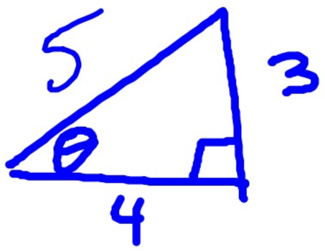


b) $\arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$



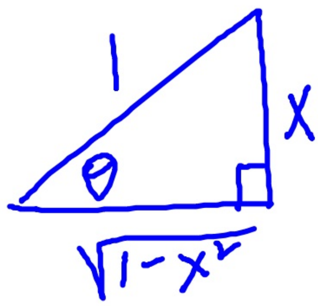
ex: Evaluate.

c) $\sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right) = \frac{3}{5}$



d) $\cos(\arcsin(x)) = \sqrt{1-x^2}$

$\cos\left(\arcsin \frac{x}{1}\right)$



Derivatives of Inverse Trigonometric Functions

THEOREM 2.18 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

***We will re-visit the proofs of these derivatives next week :)*

ex: Find the derivative.

a) $y = \sin^{-1}(2x)$

$$y' = \frac{2}{\sqrt{1-4x^2}}$$

b.) $y = \operatorname{arcsec}(e^{7x})$

$$y' = \frac{7\cancel{e^{7x}}}{\cancel{|e^{7x}|}\sqrt{e^{14x}-1}}$$

ex: Find an equation of the tangent line to the graph of f at the given point.

$$y = \arctan\left(\frac{x}{2}\right), \quad x = -2 \qquad \left(-2, \frac{-\pi}{4}\right)$$

$$y' = \frac{\frac{1}{2}}{1 + \frac{x^2}{4}}$$

$$y'(-2) = \frac{1}{4}$$

$$y + \frac{\pi}{4} = \frac{1}{4}(x+2)$$

Find the derivative: $y = \sin(\arccos x)$

$$y' = \cos(\arccos x) \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{x \cdot -1}{\sqrt{1-x^2}}$$