2.6 Derivatives of Inverse Functions

Review: Two functions are inverse functions if

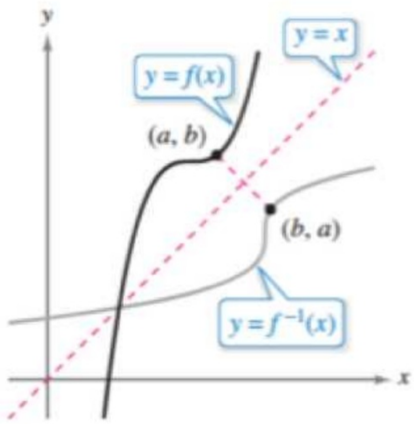
$$\boxed{\begin{gathered}\left(f \circ f^{-1}\right)(x) = x \\ \text{and} \\ \left(f^{-1} \circ f\right)(x) = x\end{gathered}}$$

Inverse Properties

- The graphs of f and f^{-1} are reflections about the line y=x.
- The domain of f is the range of f^{-1}.
- The range of f is the domain of f^{-1}.
- A function has an inverse if and only if it is one-to-one.

(monotonic)

always incr. or always decr.



THEOREM 2.17 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) =$$

$f(x) = \sqrt{x-2}$ D: $[2, \infty)$ R: $[0, \infty)$

$f'(x) = \dfrac{1}{2\sqrt{x-2}}$

$f'(6) = \dfrac{1}{4}$

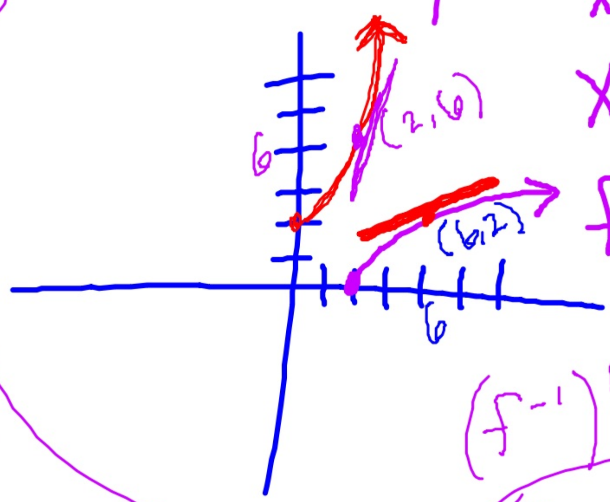
$f^{-1}(x)$ D: $[0, \infty)$ R: $[2, \infty)$

$x = \sqrt{y-2}$

$x^2 = y - 2$

$x^2 + 2 = y$

$f^{-1}(x) = x^2 + 2,$ $x \geq 0$



reciprocals

$(f^{-1})'(x) = 2x$

$(f^{-1})'(2) = 4$

ex: Find the indicated value.

c) $f(x) = x^{5/3}, \quad \left(f^{-1}\right)'(243) =$

$f(x) = x^3 + 2x - 1$

$2 = x^3 + 2x - 1$

$1 = x$

$f'(x) = 3x^2 + 2$

$f'(1) = 5$

reciprocal

$(f^{-1})'(2) = \frac{1}{5}$

$f : (1, 2)$

$f^{-1} : (2, 1)$

ex: If $f(x) = x^5 + 2x^3 + x - 1$ and

$(f \circ g)(x) = (g \circ f) = x$ find $g'(3)$ $= \dfrac{1}{12}$

inverses!

$g : (3, 1)$

$f : (1, 3)$

$3 = x^5 + 2x^3 + x - 1$

$1 = x$

$f'(x) = 5x^4 + 6x^2 + 1$

$f'(1) = 12$

reciprocal

3. 18% answered correctly

If $f(x) = x^3 + x$ and $h(x)$ is the inverse of $f(x)$, then $h'(2)$ is

A) $\frac{1}{13}$ B) $\frac{1}{4}$ C) 1 D) 4 E) 13

4. 14% answered correctly

Let f be a differentiable function such that $f(3) = 15, f'(3) = -8$, and $f'(6) = -2$, $f(6) = 3$
The function g is differentiable and $g(x) = f^{-1}(x)$ for all x. What is the value of $g'(3)$?

A) $-\frac{1}{2}$ B) $-\frac{1}{8}$ C) $\frac{1}{6}$ D) $\frac{1}{3}$ E) The value of $g'(3)$ cannot be determined from the information given.

6.

Suppose f is a one-to-one function, which is differentiable for all real numbers x. The table below gives some of the values of $f(x)$ and $f'(x)$:

x	$f(x)$	$f'(x)$
1	2	$\frac{7}{6}$
2	3	$\frac{7}{6}$
3	5	$\frac{19}{6}$
4	10	$\frac{43}{6}$

(a) Write an equation of the tangent line, T_1, to the function $f(x)$ at $x = 3$.

(b) Write an equation of the normal line, N_1, to the function $f(x)$ at $x = 3$.

reciprocal

(c) Write an equation of the tangent line, T_2, to the function $f^{-1}(x)$ at $x = 3$.

$f(2,3)$

$f^{-1}(3,2)$

$$y - 2 = \frac{6}{7}(x - 3)$$

10. Mean Score 0.95

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)dt$. Find the value $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

FR 6, 11, 17
and the Ch 2a review

26) $y = \ln|\sin x|$

$$y' = \frac{1}{\sin x}\cos x = \cot x$$

$$y'' = -\csc^2 x$$

27.) $y = \csc x$

$y' = -\csc x \cot x$

$y'' = -\left(\csc x \cdot -\csc^2 x + \cot x \cdot (-\csc x \cot x)\right)$

$= \csc^3 x + \cot^2 x \csc x$

$= \csc x(\csc^2 x + \cot^2 x)$

35.) $g(x)=\begin{cases} x^2, & x \le 2 \\ ax+b, & x>2 \end{cases}$ $g'(x)=\begin{cases} 2x, & x\le 2 \\ a, & x>2 \end{cases}$

$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) = g(2)$

$4 = 2a+b$

$4 = 8+b$

$-4 = b$

$\lim_{x\to 2^-} g'(x) = \lim_{x\to 2^+} g'(x) = g'(2)$

$4 = a$

37.) $\lim_{h \to 0} \frac{\tan(h+\frac{\pi}{4})-1}{h}$

$f(x) = \tan x$ $\qquad$ $f'(x) = \sec^2 x$

$f'\left(\frac{\pi}{4}\right) = \underline{(\sqrt{2})^2}$

2

33.) $f(x) = k - x^2$ $\quad y = -6x + 1$

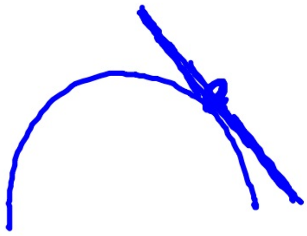
$k - x^2 = -6x + 1$

$-2x = -6$

$x = 3$

$k - 9 = -18 + 1$

$k = -8$



4.) $f(x) = \frac{x+1}{x-1}$

$\lim_{x \to 1} f(x) \quad -\infty$

20.) $g(x) = 2e^{1-x^2}$

$g'(x) = 2e^{1-x^2}(-2x)$

$\frac{d}{dx}(e^u)$

$e^u \cdot u'$

25.) $y = x^2 e^{-x}$

$$y' = x^2 \cdot e^{-x}(-1) + e^{-x}(2x)$$

$$0 = x e^{-x}(-x+2)$$

$\boxed{x=0}$ $\quad \cancel{e^{-x}=0} \quad$ $-x+2=0$

$\boxed{x=2}$

$(0,0)$ $\quad$ $\left(2, \frac{4}{e^2}\right)$

18.) $s(x) = \frac{\ln x^2}{x}$

$$s(x) = \frac{2\ln x}{x}$$

$$s'(x) = 2\left(\frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}\right) \qquad x = e$$

$$s'(x) = \frac{2(1-\ln x)}{x^2} \qquad 2(1-\ln x) = 0$$

$$1 - \ln x = 0$$

$$e^1 = e^{\ln x}$$

28.) $y = (x^2+1)^5$

$y' = 5(x^2+1)^4 \cdot 2x$

$y' = 10x(x^2+1)^4$

$y'' = 10x \cdot 4(x^2+1)^3 2x + (x^2+1)^4 \cdot 10$

$y'' = 10(x^2+1)^3 \left[8x^2 + x^2+1 \right]$

$= 10(x^2+1)^3(9x^2+1)$

$\frac{1}{x-2} = \frac{-1}{2-x}$

$\ln 2^4$

$4 \ln 2$

$\ln 16$

22.) $g(x) = \sec(5x) \quad c = \frac{\pi}{3}$

$$g'(x) = \sec(5x)\tan(5x) \cdot 5$$

$$g'\left(\frac{\pi}{3}\right) = \sec\frac{5\pi}{3}\tan\frac{5\pi}{3}(5)$$

$$(+2)(-\sqrt{3})(5)$$

$$-10\sqrt{3}$$

FR 6.) skip e

17b.)

$f(x) = \sqrt{x^4 - 16x^2}$

$f(-x) = \sqrt{x^4 - 16x^2}$ (even; symm. with y-axis)

11c.) $f(x) = -2 + \ln(x^2) = -2 + 2\ln x$

$x = 1$ $\quad f'(x) = 0 + 2 \cdot \frac{1}{x}$

$(1, -2)$ $\quad f'(1) = 2$

$$\boxed{y + 2 = 2(x - 1)}$$