2.4 Chain Rule

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

THEOREM 2.11 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$f'(g(x)) \cdot g'(x)$$

$$x^2$$
$$2x$$

$$\frac{d}{dx}[f(g(x))] =$$

$$y = (3x-2)^2$$
$$y = 9x^2 - 12x + 4$$
$$y' = 18x - 12$$
$$= 6(3x-2)$$

$$y = (3x-2)^2$$
$$y' = 2(3x-2)^1 \cdot 3$$
$$y' = 6(3x-2)$$

ex: Which function(s) are good candidates for the chain rule?

- ✓ $y = \sin\left(x^2 + 1\right)$
- ✓ $y = e^{2x}$
- ✓ $y = \ln\left(x - 5\right)$
- $f\left(x\right) = \sin x \cos x$
- ✓ maybe $y = \dfrac{5}{x+1} = 5(x+1)^{-1}$

ex: Find the derivative.

a) $y = \sin 2x$

$y' = \cos 2x \cdot 2$

$= 2\cos 2x$

b) $y = \sin(\cos x)$

$y' = \cos(\cos x) \cdot (-\sin x)$

$= -\cos(\cos x)\sin x$

ex: Find the derivative.

c) $y = \sqrt{e^x + 1} = (e^x + 1)^{1/2}$

$y' = \frac{1}{2}(e^x + 1)^{-1/2} \cdot e^x$

$(x)^3$

$3(x)^2$

d) $y = \cos^4 x = (\cos x)^4$

$y' = 4(\cos x)^3(-\sin x)$

ex: Find the derivative.

e) $y = \tan^2 x$ $= (\tan x)^2$

$y' = 2(\tan x)^1 \sec^2 x$

f) $y = \tan x^2$ $= \tan(x^2)$

$y' = \sec^2(x^2) \cdot 2x$

ex: Find the derivative.

g) $y = \sec 3x$

$y = \sec x$

$y' = \tan x \sec x$

$y' = \tan 3x \sec 3x \cdot 3$

ex: Find the derivative.

j) $f(x) = x^2\sqrt{1-x^2}$ $= x^2(1-x^2)^{1/2}$

$$f'(x) = \underline{x^2 \cdot \tfrac{1}{2}(1-x^2)^{-1/2}(-2x)} + (1-x^2)^{1/2} \cdot 2x$$

$$= \frac{-x^3}{(1-x^2)^{1/2}} + \frac{2x(1-x^2)^{1/2}}{1} \frac{(1-x^2)^{1/2}}{(1-x^2)^{1/2}}$$

$$= \frac{-x^3 + 2x(1-x^2)}{(1-x^2)^{1/2}}$$

$$= \frac{-3x^3 + 2x}{(1-x^2)^{1/2}}$$

ex: Find the derivative.

k) $y = \cos^4(3x) = (\cos 3x)^4$

$y' = 4(\cos 3x)^3(-\sin 3x \cdot 3)$

$= -12\cos^3 3x \sin 3x$

$y = \sin x$

$y' = \cos x \cdot 1$

$y = \cos 3x$

$y' = -\sin 3x \cdot 3$

$= -3\sin 3x$

ex: $g(x) = \tan x$, $\quad \dfrac{d^2y}{dx^2} = ?$

$$g'(x) = \sec^2 x = (\sec x)^2$$

$$g''(x) = 2(\sec x)^1 \cdot \sec x \tan x$$

$$= 2\sec^2 x \tan x$$

ex: Write the equation of the tangent line to

$$y=\left(\frac{1}{x^2+9}\right)^2 \text{ at } x=\left(1,\tfrac{1}{100}\right) \rightarrow (x^2+9)^{-2}$$

$$y' = -2(x^2+9)^{-3}(2x)$$

$$y'(1) = \frac{-1}{250}$$

$$y - \frac{1}{100} = \frac{-1}{250}(x-1)$$

ex: Find the equation of the line that is tangent to $y = \sqrt{3x-1}$ and perpendicular to $3y + 2x = 3$. $\leftarrow$ $m = -\frac{2}{3}$

normal

$\left(\frac{2}{3}, 1\right)$ $\perp$: $\frac{3}{2}$
m

$y' = \frac{1}{2}(3x-1)^{-1/2} \cdot 3$

$y' = \frac{3}{2\sqrt{3x-1}}$

$\cancel{\frac{3}{2}} = \frac{\cancel{3}}{\cancel{2}\sqrt{3x-1}}$

$\sqrt{3x-1} = 1$

$x = 2/3$

$$y - 1 = \frac{3}{2}\left(x - \frac{2}{3}\right)$$

ex:

$f'(g(x))\cdot g'(x)$

$f'(g(0))g'(0)$

$f'(2)(5)$

$3\cdot 5$

15

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	–1	2	5
1	–1	2	4	0
2	7	3	11	0.5

1. **Based on the values in the table above.**
   If $K(x) = f(g(x))$, then $K'(0) =$

   (A) 15 (B) 35 (C) –5 (D) –1 (E) 7

2. Let $g(x) = x\sin(x^2)$. Then $g'(x) =$

(A) $2x^2\cos(x^2)$ (B) $-2x^2\cos(x^2)$ (C) $2x^3\cos(x^2)$
D) $2x^2\cos(x^2) + \sin(x^2)$ (E) $-2x^2\cos(x^2) + \sin(x^2)$

ex: Find the derivative.

a) $y = \tan x \cot x$

b) $y = \dfrac{2}{(x-3)^4}$

c) $y = \left(\dfrac{x}{x^2+1}\right)^{-1}$ $\dfrac{x^2+1}{x}$

d) $y = \dfrac{\tan x}{\sin x - 1}$

e) $y = 2\sin x \cos x$

f) $y = \cot x \csc x$

ex: Find the indicated derivative at the given point using your calculator.

a) $\frac{d}{dx}(\ln x)\Big|_{x=8}$

b) $f(x) = \tan^{-1} x, \quad f'(5)$

c) $y = \frac{3x+2}{\sqrt{x^2+5}}, \quad \frac{d^2y}{dx^2}\Big|_{x=\frac{1}{2}}$