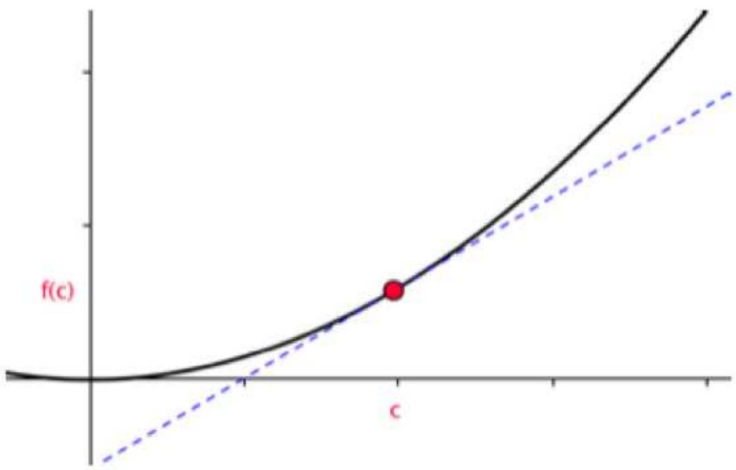
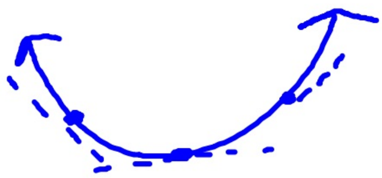
2.1 Definition Of A Derivative

$y = 7x - 4$
$m = 7$

The Tangent Line Problem

Task: Write the equation of the tangent line to f(x) at x=c.





DEFINITION OF TANGENT LINE WITH SLOPE m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \boxed{\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m}$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

*The slope of the tangent line to the graph of f at the point (c, f(c)) with slope m is also known as the <u>slope of the graph of f at x=c.</u>

ex: Find the slope of the tangent line at the given point.

a) $f(x) = x^2 + 1, \quad (2,5)$

$c = 2$

$f(2+h) = (2+h)^2 + 1$

$f(2) = 5$

$$m = \lim_{h \to 0} \frac{4 + 4h + h^2 + 1 - 5}{h}$$

$$= \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} (4 + h) = 4$$

At the point (2, 5) the slope of the tangent line for f(x) is 4.

ex: Find the slope of the tangent line at the given point.

b) $f(x) = x^2 + 1, \quad (-1,2)$

$$m = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 - 2h + \cancel{1} + \cancel{1} - \cancel{2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(h-2)}{\cancel{h}} = -2$$

ex: Find the slope of the tangent line at the given point.

c) $f(x) = x^2 + 1, \quad (0,1)$

$m = 0$

ex: Find the slope of the tangent line at the given point.

d) $f(x) = 3, \quad (21,3)$

$c = 21$

$f(21+h) = 3$

$$m = \lim_{h \to 0} \frac{3-3}{h}$$

$$\lim_{h \to 0} \frac{0}{h} \qquad \frac{0}{-.01} \qquad \frac{0}{.01}$$

$m = 0$

ex: Interpret the expression.

$$\lim_{h \to 0} \frac{f(6+h) - f(6)}{h} = 100$$

The slope of the line tangent to f(x) is 100 at x = 6 (or (6, f(6)).

Derivative - a formula used to find the slope of a tangent line

Vocab:

- differentiation - the process of finding a derivative
- differentiate - to find a derivative
- differntiable - a derivative exists

not on exam

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notation for derivatives

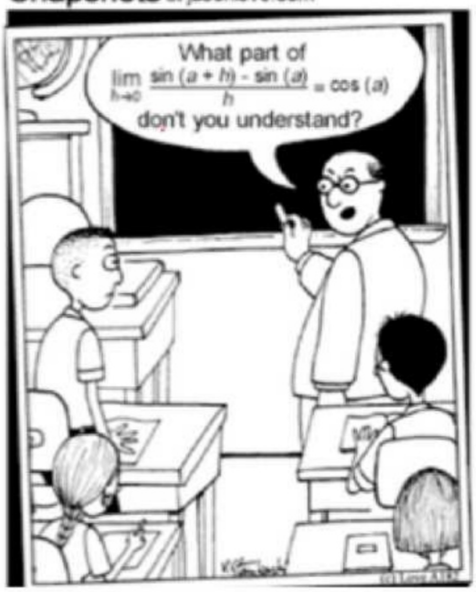
DEFINITION OF THE DERIVATIVE OF A FUNCTION

The derivative of f at x is given by

$$\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

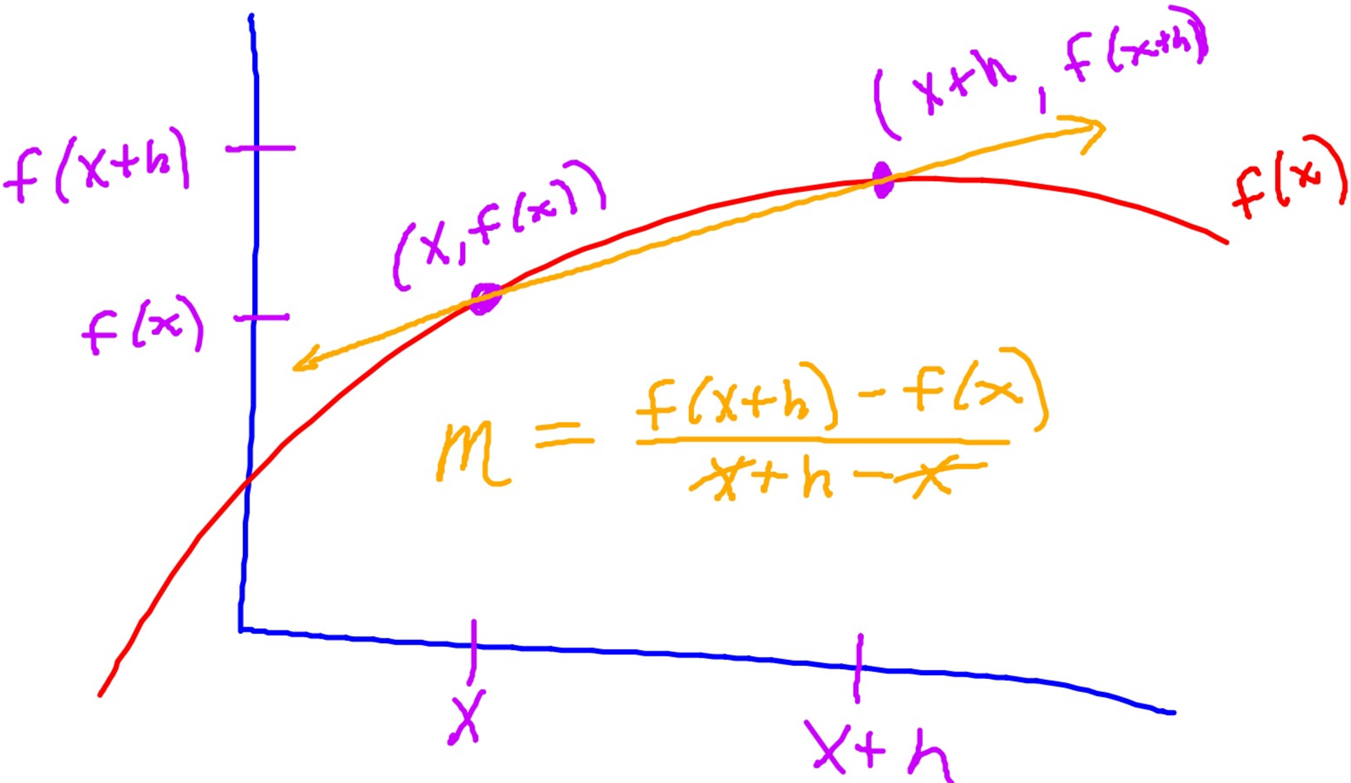
provided the limit exists. For all x for which this limit exists, f' is a function of x.

Snapshots at jasonlove.com



Derivative Synonyms:

- rate of change
- slope of a tangent line



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

ex: Find the derivative using the limit process.

a) $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$f'(x) = 2x$$

ex: Find the derivative using the limit process.

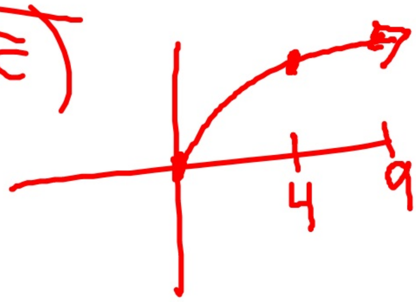
b) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \overset{1}{\cancel{h}} - \cancel{x}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

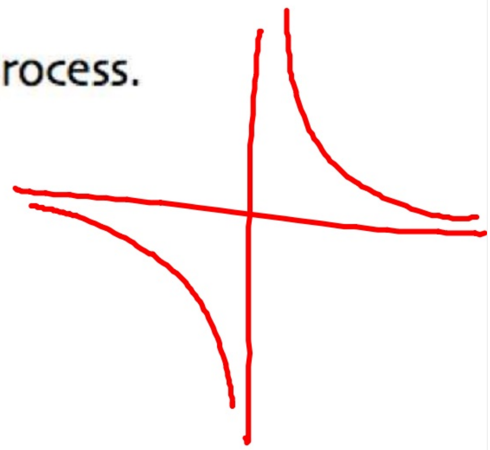
$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$



ex: Find the derivative using the limit process.

c) $f(x)=\dfrac{1}{x}$



$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\lim_{h\to 0}\frac{x-(x+h)}{x(x+h)}\cdot\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{-\cancel{h}}{x\cancel{h}(x+h)}=\frac{-1}{x^2}$$

$$f'(3)=-\frac{1}{9}$$

ex: Find the slope at the given point.

a) $f(x) = \dfrac{1}{x}, \quad \left(10, \dfrac{1}{10}\right)$

ex: Find the slope at the given point.

b) $f(x) = \sqrt{x}, \quad (0,0)$

ex: Interpret the expression.

$$g'(22) = 5$$

The slope of the tangent line for g(x) at x = 22 is 5.

OR

The slope for g(x) at x = 22 is 5.

- Alternate Form Of The Derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

ex: Use the alternate form to find the slope of the curve at the given x-value.

a) $f(x) = x^2 + 1, \quad (2,5)$

$$f'(2) = \lim_{x \to 2} \frac{x^2 + 1 - 5}{x - 2}$$

$$= \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

$$= \lim_{x \to 2} (x+2) = 4$$

ex: Use the alternate form to find the slope of the curve at the given x-value.

b) $f(x) = 17 - 4x, \quad (1,13)$

$$f'(1) = \lim_{x \to 1} \frac{17 - 4x - 13}{x - 1}$$

$$= \lim_{x \to 1} \frac{4 - 4x}{x - 1} = \lim_{x \to 1} \frac{-4\cancel{(x-1)}}{\cancel{x-1}}$$

$$f'(1) = -4$$

ex: Use the alternate form to find the slope of the curve at the given x-value.

$f(x) = |x|$

$f'(x) = \frac{|x|}{x}$

c) $f(x) = |x|, \quad (0,0)$

$$f'(0) = \lim_{x \to 0} \frac{|x| - 0}{x - 0}$$

$$= \lim_{x \to 0} \frac{|x|}{x}$$

dne

$\therefore$ $f'(0)$ does not exist

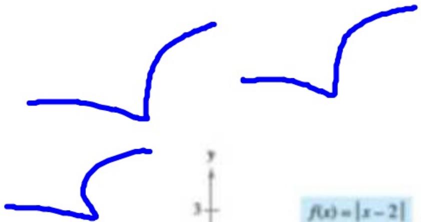
ex: Use the alternate form to find the slope of the curve at the given x-value.

d) $f(x) = \sqrt[3]{x}, \quad (0,0)$

ex: Use the alternate form to find the slope of the curve at the given x-value.

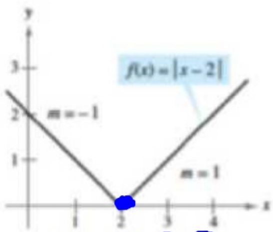
e) $f(x) = \sqrt{x}, \quad (0,0)$

A function f is **NOT DIFFERENTIABLE** at...

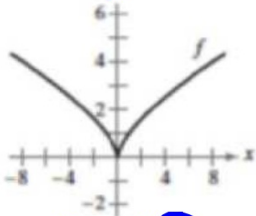


$x^{2/3}$

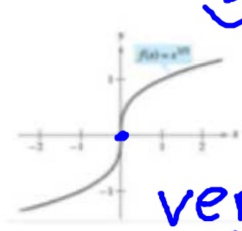
$\sqrt[3]{x}$



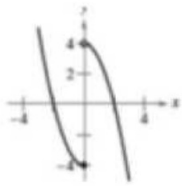
corner

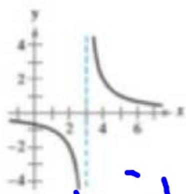


cusp



vertical tangent





any discontinuity

If f is differentiable at x=c, then f must be continuous at x=c.

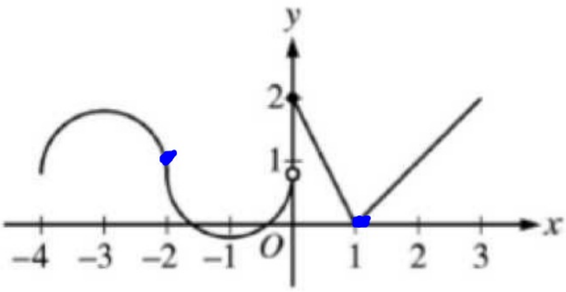
BUT...

If f is continuous at x=c, f may or MAY NOT be differentiable at x=c.

MORAL OF THE STORY:

DIFFERNTIABILITY IMPLIES CONTINUITY

ex:



Graph of f

The graph of the piecewise-defined function f is shown in the figure above. The graph has a vertical tangent line at $x = -2$ and horizontal tangent lines at $x = -3$ and $x = -1$. What are all values of x, $-4 < x < 3$, at which f is continuous but not differentiable?

(A) $x = 1$

(B) $x = -2$ and $x = 0$

(C) $x = -2$ and $x = 1$

(D) $x = 0$ and $x = 1$

ex: The limit below represents the derivative of f at x=c for a function f and a number c. Find f and c.

a) $$\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}$$

$f(x) = \sin x$

$c = \pi/2$

$f'(\frac{\pi}{2})$

b) $$\lim_{x\to e}\frac{\ln x-1}{x-e}$$

$f(x) = \ln x$ $\quad f'(e)$

$c = e$

ex:

What is $\lim_{h \to 0} \frac{\cos\left(\frac{3\pi}{2} + h\right) - \cos\left(\frac{3\pi}{2}\right)}{h}$?

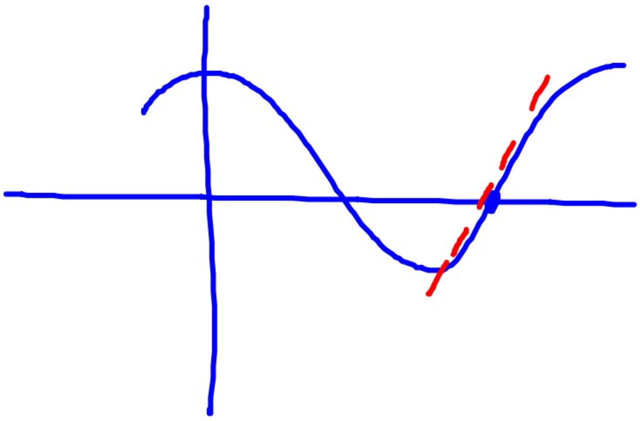
(A) 1

(B) $\frac{\sqrt{2}}{2}$

~~(C) 0~~

~~(D) −1~~

~~(E)~~ The limit does not exist.



$f(x) = \cos x$

$c = \frac{3\pi}{2}$