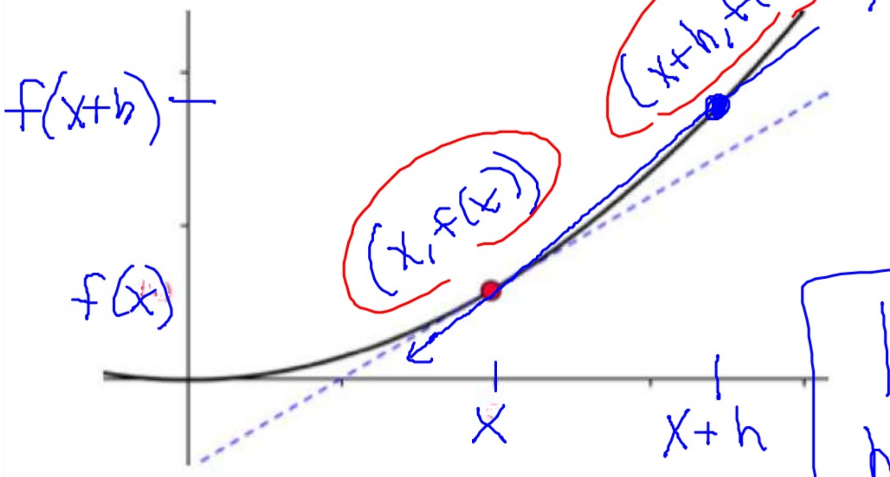
2.1 Definition Of A Derivative

The Tangent Line Problem

Task: Write the equation of the tangent line to f(x) at x=c.



$$m = \frac{f(x+h)-f(x)}{x+h-x}$$

$$m = \frac{f(x+h)-f(x)}{h}$$

$$\boxed{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

A	16
B	13
C	3
D	1
F	0

DEFINITION OF TANGENT LINE WITH SLOPE m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m \qquad \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

*The slope of the tangent line to the graph of f at the point (c, f(c)) with slope m is also known as the <u>slope of the graph of f at x=c.</u>

ex: Interpret the expression.

$$\lim_{h \to 0} \frac{f(6+h) - f(6)}{h} = 100$$

$f'(6) = 100$

The slope of f(x) at x = 6 is 100

The derivative of f(x) at x = 6 is 100.

Derivative - a formula used to find the slope of a tangent line

Vocab:

- differentiation - the process of finding a derivative
- differentiate - to find a derivative
- differentiable - a derivative exists

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notation for derivatives

not on AP

DEFINITION OF THE DERIVATIVE OF A FUNCTION

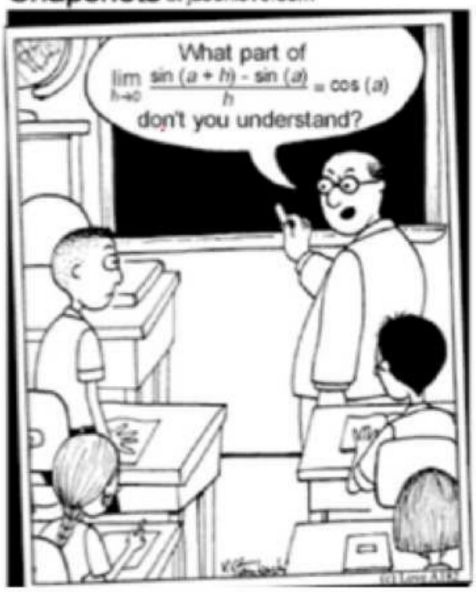
$f'(x)$

The **derivative** of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Snapshots at jasonlove.com



Derivative Synonyms:
- rate of change
- slope of a tangent line
- slope

ex: Find the derivative using the limit process.

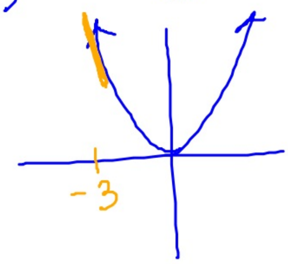
a) $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h)$$

$$\boxed{f'(x) = 2x}$$

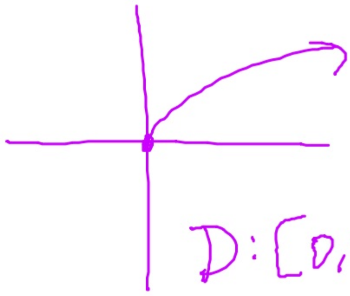
$$f'(-3) = -6$$



ex: Find the derivative using the limit process.

b) $f(x)=\sqrt{x}$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$



D: $[0,\infty)$

$$f'(x)=\frac{1}{2\sqrt{x}}$$

Domain of $f'(x)$

$(0,\infty)$

ex: Find the derivative using the limit process.

c) $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x-(x+h)}{x(x+h)} \cdot \frac{1}{h}$$

$$f'(x) = -\frac{1}{x^2}$$

ex: Find the slope at the given point.

a) $f(x) = \frac{1}{x}, \quad \left(10, \frac{1}{10}\right)$ $\quad f'(x) = -\frac{1}{x^2}$

$f'(10) = -\frac{1}{100}$

ex: Interpret the expression.

$$g'(22) = 5$$

The slope at x = 22 for g(x) is 5.

The rate of change at x = 22 for g(x) is 5.

- Alternate Form Of The Derivative

This form will give you the slope value at x = c.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$f(x) = x^2$

$f'(-3) = ?$

$$f'(-3) = \lim_{x \to -3} \frac{x^2 - 9}{x + 3}$$

$$= \lim_{x \to -3} (x - 3) = -6$$

ex: Use the alternate form to find the slope of the curve at the given x-value.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

a) $f(x) = x^2 + 1, \quad (2, 5)$

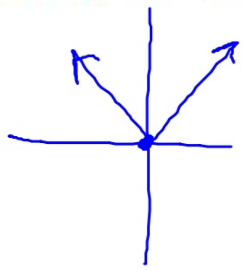
$f'(2) =$ ______

$$f'(2) = \lim_{x \to 2} \frac{x^2 + 1 - 5}{x - 2}$$

$$= \lim_{x \to 2} (x + 2)$$

$$= 4$$

ex: Use the alternate form to find the slope of the curve at the given x-value.



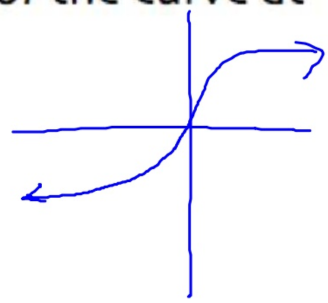
c) $f(x)=|x|, \quad (0,0)$

$$f'(0)=\lim_{x\to 0}\frac{|x|}{x} \quad \text{dne}$$

$$\lim_{x\to 0^-}\frac{|x|}{x} \neq \lim_{x\to 0^+}\frac{|x|}{x}$$

$$-1 \neq 1$$

ex: Use the alternate form to find the slope of the curve at the given x-value.



d) $f(x) = \sqrt[3]{x}, \quad (0,0)$

$$f'(0) = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x - 0}$$

$$= \lim_{x \to 0} \frac{1}{x^{2/3}}$$

$$\lim_{x \to 0^-} \frac{1}{x^{2/3}} = \frac{1}{(-.01)^{2/3}} = \frac{1}{.00001} = \text{Big}$$

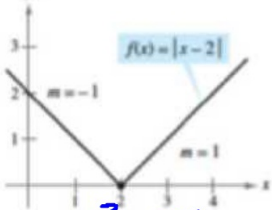
$$\lim_{x \to 0^+} \frac{1}{x^{2/3}} = \infty$$

∞

slope does not exist

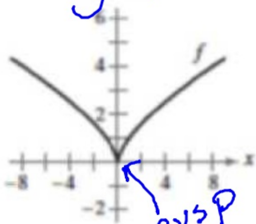
A function f is NOT DIFFERENTIABLE at...

$y = |x-2|$



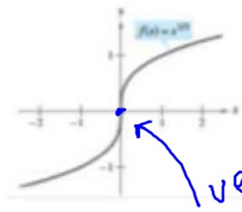
corner

$y = x^{2/3}$

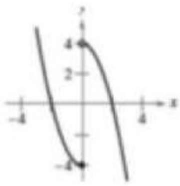


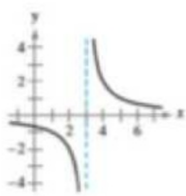
cusp

$y = \sqrt[3]{x}$



vertical tangent





any type of discontinuity

If f is differentiable at x=c, then f must be continuous at x=c.

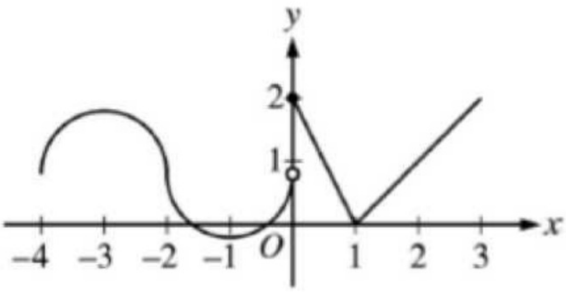
BUT...

If f is continuous at x=c, f may or MAY NOT be differentiable at x=c.

MORAL OF THE STORY:

DIFFERNTIABILITY IMPLIES CONTINUITY

ex:



Graph of f

The graph of the piecewise-defined function f is shown in the figure above. The graph has a vertical tangent line at $x = -2$ and horizontal tangent lines at $x = -3$ and $x = -1$. What are all values of x, $-4 < x < 3$, at which f is continuous but not differentiable?

(A) $x = 1$

(B) $x = -2$ and $x = 0$

(C) $x = -2$ and $x = 1$

(D) $x = 0$ and $x = 1$

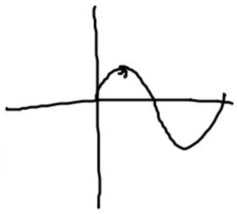
ex: The limit below represents the derivative of f at x=c for a function f and a number c. Find f and c.

a) $$\lim_{h \to 0} \frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}$$

$f(x) = \sin x$

$c = \frac{\pi}{2}$

$f'(\frac{\pi}{2}) = 0$



b) $$\lim_{x \to e} \frac{\ln x - 1}{x - e}$$

$f(x) = \ln x$

$c = e$

ex:

What is $\lim\limits_{h \to 0} \dfrac{\cos\left(\frac{3\pi}{2} + h\right) - \cos\left(\frac{3\pi}{2}\right)}{h}$?

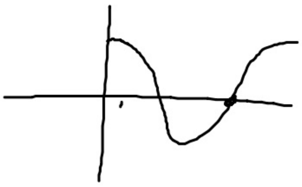
(A) 1

(B) $\frac{\sqrt{2}}{2}$

(C) 0

(D) -1

(E) The limit does not exist.



$$\lim_{h \to 0} \frac{\overset{(0)}{\cos\frac{3\pi}{2}}\cos h - \overset{(-1)}{\sin\frac{3\pi}{2}}\sin h - 0}{h}$$

$$\lim_{h \to 0} \frac{0 + \sin h}{h} = 1$$