A2 Reference Sheet

Number Sets

Number Set	Symbol	Definition
Real	$\mathbb{R}$	A **real number** is a value that can be represented as a quantity on a continuous number line.
Rational	$\mathbb{Q}$	A **rational number** is any quantity that can be expressed as the ratio of two integers. Ex: 4(since $4=\frac{8}{2}$), 1.2 (since $1.2=\frac{12}{10}=\frac{6}{5}$), $-\sqrt{9}$ (since $-\sqrt{9}=-3=\frac{-3}{1}$, *etc.*
Integers	$\mathbb{Z}$	The set of **integers** contains whole numbers, negative whole numbers and zero. Z={…-3, -2, -1, 0, 1, 2, 3…}
Whole	W	**Whole** numbers are nonnegative integers W={0, 1, 2, 3…}
Natural	$\mathbb{N}$	**Natural** numbers are positive integers. This set is commonly referred to as the "counting" numbers set. N={1, 2, 3…}
Digits	D	A **digit** is any number that can be found in a phone number. D={0, 1, 2…9}
Irrational	I	An **irrational** number is any quantity that can NOT be expressed as a fraction (any nonrepeating & nonterminating decimal) Ex: π, $\sqrt{2}$
Transcendental	T	**Transcendental** numbers are numbers that are NOT the solution to an algebraic equation. Ex: π, ϕ (phi – the golden number), e

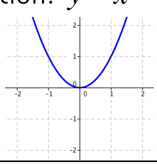
Quadratic Functions

3 Forms

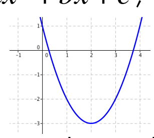
Form	Equation	
Standard Form	$y=ax^2+bx+c$	• The x-coordinate of the vertex is $x=-b/2a$ • The axis of symmetry is $x=-b/2a$
Vertex Form	$y=a(x-h)^2+k$	• The vertex is (h,k)
Intercept Form	$y=a(x-p)(x-q)$	• The x-intercepts are $x=p$ and $x=q$. • The x-coordinate of the vertex is the average of the x-intercepts.

Graphing Quadratics – plot the vertex and two other points, one to the left and one to the right of the vertex.

Parent Function: $y=x^2$

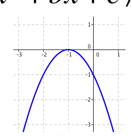


$y=ax^2+bx+c$, $a>0$



MINIMUM value at the vertex

$y=ax^2+bx+c$, $a<0$



MAXIMUM value at the vertex

Complex Numbers: $a+bi$ *(standard form)*

Imaginary Numbers:

$\sqrt{-1}=i$

Powers of i

$i^1=i$

$i^2=-1$

$i^3=-i$

$i^4=1$

Rationalizing with Complex Numbers

Ex: $\frac{3+i}{7i}$

*MULTIPLY the numerator and denominator by i

$$\frac{3+i}{7i}\cdot\frac{i}{i}$$
$$\frac{3i+i^2}{7i^2}$$
$$\frac{3i-1}{-7}$$
$$\frac{1}{7}-\frac{3}{7}i$$

Ex: $\frac{3+i}{7-i}$

*MULTIPLY the numerator and denominator by the conjugate of the denominator, $7+i$

$$\frac{3+i}{7-i}\cdot\frac{7+i}{7+i}$$
$$\frac{21+10i+i^2}{49-i^2}$$
$$\frac{21+10i-1}{49+1}$$
$$\frac{20+10i}{50}$$
$$\frac{2}{5}+\frac{1}{5}i$$

Completing the Square

To complete the square: $\left(\frac{b}{2a}\right)^2$	**Use completing the square to…**

Use completing the square to…

- Rewrite a quadratic function from standard form to vertex form.

$$y = 2x^2 + 12x + 10$$
$$y = 2\left(x^2 + 6x \_\_\_\_\right) \_\_\_\_ + 10$$
$$y = 2\left(x^2 + 6x + 9\right) - 18 + 10$$
$$y = 2\left(x + 3\right)^2 - 8$$

- Solve quadratic equations in standard form.

$$x^2 - 4x + 5 = 0$$
$$\left(x^2 - 4x \_\_\_\_\right) \_\_\_\_ + 5 = 0$$
$$\left(x^2 - 4x + 4\right) - 4 + 5 = 0$$
$$\left(x - 2\right)^2 + 1 = 0$$
$$\left(x - 2\right)^2 = -1$$
$$\sqrt{\left(x - 2\right)^2} = \sqrt{-1}$$
$$\left|x - 2\right| = i$$
$$x - 2 = \pm i$$
$$x = 2 \pm i$$

4 Methods to Solve Quadratic Equations

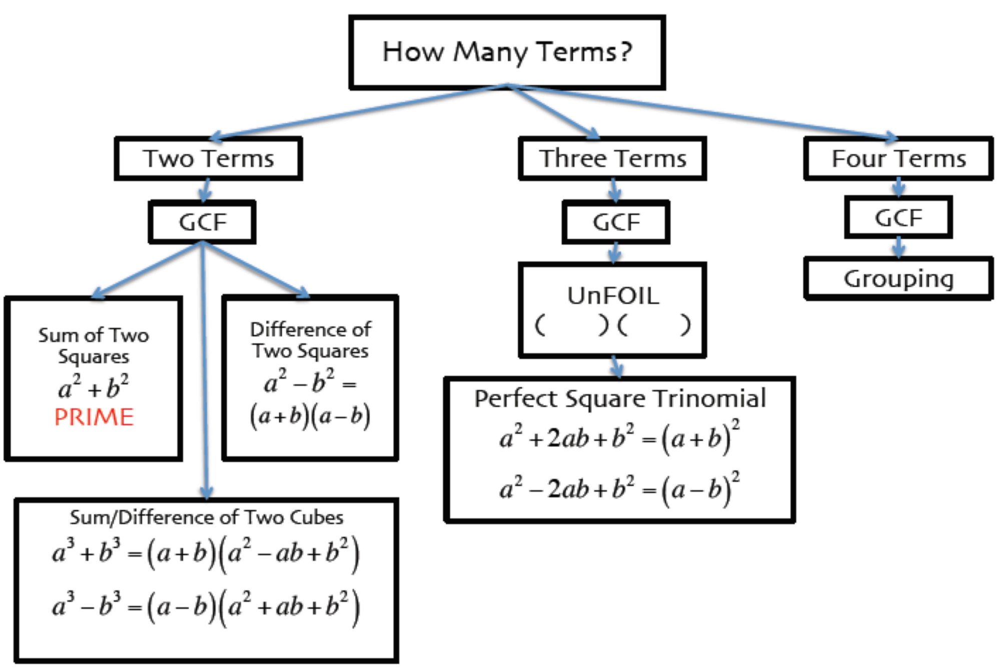
<u>Factoring</u> – Use when $ax^2 + bx + c = 0$ and ac has a set of factors that sum to b	<u>Square Roots</u> - Use when $ax^2 + c = 0$ or $a\left(x - h\right)^2 + k = 0$	<u>Completing the Square</u> – Use when $ax^2 + bx + c = 0$ (*Convenient when $\frac{b}{a}$ is *even*)	<u>Quadratic Formula</u> – Use when $ax^2 + bx + c = 0$ (*Convenient when a, b, c are *small*.) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The discriminant: $b^2 - 4ac$

*Use the discriminant to determine the number and type of roots(solutions) of a quadratic equation.

If $b^2 - 4ac > 0$,	If $b^2 - 4ac = 0$,	If $b^2 - 4ac < 0$,
• Number of Solutions: 2 distinct • Type of Solutions: real • The graph of $y = ax^2 + bx + c$ has <u>two</u> x-intercepts.	• Number of Solutions: 1 repeated • Type of Solutions: real • The graph of $y = ax^2 + bx + c$ has <u>one</u> x-intercept.	• Number of Solutions: 2 distinct • Type of Solutions: imaginary • The graph of $y = ax^2 + bx + c$ has <u>NO</u> x-intercepts.

Factoring



Polynomial Functions

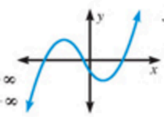
Exponent Properties

Property Name	Definition	Example
Product of Powers	$a^m \cdot a^n = a^{m+n}$	$5^3 \cdot 5^{-1} = 5^{3+(-1)} = 5^2 = 25$
Power of a Power	$(a^m)^n = a^{mn}$	$(3^3)^2 = 3^{3 \cdot 2} = 3^6 = 729$
Power of a Product	$(ab)^m = a^m b^m$	$(2 \cdot 3)^4 = 2^4 \cdot 3^4 = 1296$
Negative Exponent	$a^{-m} = \frac{1}{a^m}, a \neq 0$	$7^{-2} = \frac{1}{7^2} = \frac{1}{49}$
Zero Exponent	$a^0 = 1, a \neq 0$	$(-89)^0 = 1$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{6^{-3}}{6^{-6}} = 6^{-3-(-6)} = 6^3 = 216$
Power of a Quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$	$\left(\frac{4}{7}\right)^2 = \frac{4^2}{7^2} = \frac{16}{49}$

End Behavior

Degree: odd
Leading coefficient: positive

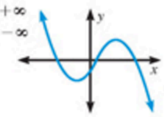
$f(x) \to -\infty$ as $x \to -\infty$



$f(x) \to +\infty$ as $x \to +\infty$

Degree: odd
Leading coefficient: negative

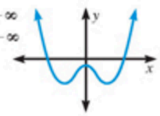
$f(x) \to +\infty$ as $x \to -\infty$



$f(x) \to -\infty$ as $x \to +\infty$

Degree: even
Leading coefficient: positive

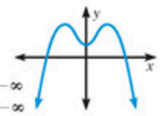
$f(x) \to +\infty$ as $x \to -\infty$



$f(x) \to +\infty$ as $x \to +\infty$

Degree: even
Leading coefficient: negative

$f(x) \to -\infty$ as $x \to -\infty$



$f(x) \to -\infty$ as $x \to +\infty$

Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

Factor Theorem

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

The Rational Zero Theorem

If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ has *integer* coefficients, then every rational zero of f has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

Finding Zeros Using The Rational Zero Theorem

Find all real zeros of $f(x) = x^3 - 8x^2 + 11x + 20$.

Solution

STEP 1 **List** the possible rational zeros. The leading coefficient is 1 and the constant term is 20. So, the possible rational zeros are:

$$x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1}, \pm\frac{5}{1}, \pm\frac{10}{1}, \pm\frac{20}{1}$$

STEP 2 **Test** these zeros using synthetic division.

Test $x = 1$:

1	1	−8	11	20
		1	−7	4
	1	−7	4	**24**

1 is not a zero.

Test $x = -1$:

−1	1	−8	11	20
		−1	9	−20
	1	−9	20	**0**

−1 is a zero.

Because −1 is a zero of f, you can write $f(x) = (x + 1)(x^2 - 9x + 20)$.

STEP 3 **Factor** the trinomial in $f(x)$ and use the factor theorem.

$$f(x) = (x + 1)(x^2 - 9x + 20) = (x + 1)(x - 4)(x - 5)$$

▸ The zeros of f are −1, 4, and 5.

Rational Exponents and Radical Functions

Rational Exponents

$$\sqrt[n]{a^m} = a^{m/n}$$

ex: Evaluate.

$$4^{5/2} = \left(\sqrt{4}\right)^5 = (2)^5 = 32$$ *(easiest to take the root first!)*

ex: Simplify.

a) $\sqrt[3]{135} = \sqrt[3]{27 \cdot 5} = 3\sqrt[3]{5}$ *(look for perfect powers)*

b) $\frac{\sqrt[5]{2}}{\sqrt[5]{9}} = \frac{\sqrt[5]{2}}{\sqrt[5]{3^2}} \cdot \frac{\sqrt[5]{3^3}}{\sqrt[5]{3^3}} = \frac{\sqrt[5]{54}}{3}$

Simplifying Roots

	Rule	Example
When n is odd	$\sqrt[n]{x^n} = x$	$\sqrt[3]{64x^3y^5} = 4xy\sqrt[3]{y^2}$
When n is even	$\sqrt[n]{x^n} = \lvert x \rvert$	$\sqrt[4]{32xy^4x^8} = 2x^2\lvert y \rvert\sqrt[4]{2x}$

Function Operations

Operation	Definition	Example: $f(x) = x^2 - 3$, $g(x) = x + 2$
Addition	$(f+g)(x) = f(x) + g(x)$	$(f+g)(x) = x^2 - 3 + x + 2 = x^2 + x - 1$
Subtraction	$(f-g)(x) = f(x) - g(x)$	$(f-g)(x) = x^2 - 3 - (x+2) = x^2 - x - 5$
Multiplication	$(fg)(x) = f(x)g(x)$	$(fg)(x) = (x^2 - 3)(x + 2) = x^3 + 2x^2 - 3x - 6$
Division	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$\left(\frac{f}{g}\right)(x) = \frac{x^2 - 3}{x + 2}$
Compositions	$(f \circ g)(x) = f(g(x))$ $(g \circ f)(x) = g(f(x))$	$(f \circ g)(x) = f(x+2) = (x+2)^2 - 3 = x^2 + 4x + 1$ $(g \circ f)(x) = g(x^2 - 3) = x^2 - 3 + 2 = x^2 - 1$

Inverse Functions

Properties of Inverse Functions

If $f(x)$ and $g(x)$ are inverse functions then…

- $(f \circ g)(x) = x$ AND $(g \circ f)(x) = x$
- The graphs of $f(x)$ and $g(x)$ are reflections about the line $y = x$
- The domain of $f(x)$ is the range of $g(x)$.
- The range of $f(x)$ is the domain of $g(x)$.
- If $f(x)$ contains the point (a, b) then $g(x)$ contains the point (b, a)

Verify that $f(x) = 3x - 5$ and $f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}$ are inverse functions.

Solution

STEP 1 Show that $f(f^{-1}(x)) = x$.

$f(f^{-1}(x)) = f\left(\frac{1}{3}x + \frac{5}{3}\right)$

$= 3\left(\frac{1}{3}x + \frac{5}{3}\right) - 5$

$= x + 5 - 5$

$= x$ ✓

STEP 2 Show that $f^{-1}(f(x)) = x$.

$f^{-1}(f(x)) = f^{-1}(3x - 5)$

$= \frac{1}{3}(3x - 5) + \frac{5}{3}$

$= x - \frac{5}{3} + \frac{5}{3}$

$= x$ ✓

To find an Inverse:

Ex: $f(x) = x^3 + 4$

1. switch x and y	$x = y^3 + 4$
2. solve for y	$y = \sqrt[3]{x - 4}$
3. label appropriately	$f^{-1}(x) = \sqrt[3]{x - 4}$

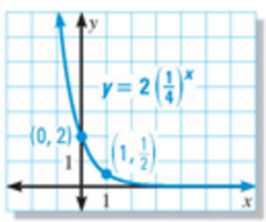
Exponential and Logarithmic Functions

Exponential Functions: $y = ab^x$ $(a \neq 0, b > 0, b \neq 1)$

"b" is the growth/decay factor

If $0 < b < 1$, then $y = ab^x$ represents exponential <u>decay</u>.

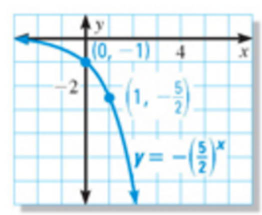
Ex: $y = 2\left(\frac{1}{4}\right)^x$



*notice the right side of the graph APPROACHES the asymptote

If $b > 1$, then $y = ab^x$ represents exponential <u>growth</u>.

Ex: $y = -\left(\frac{5}{2}\right)^x$



*notice the right side of the graph MOVES AWAY from the asymptote

Solving Exponential Equations

Cases	*Make the bases equal*	*Can't make the bases equal*	*Quadratic Form*
	$9^{x-2} = 27^{2x-5}$ $(3^2)^{x-2} = (3^3)^{2x-5}$ $3^{2x-4} = 3^{6x-15}$ $2x - 4 = 6x - 16$ $12 = 4x$ $x = 3$	$2 \cdot 5^{x+4} - 9 = -3$ $5^{x+4} = 3$ $\log_5(5^{x+4}) = \log_5(3)$ $x + 4 = \log_5 3$ $x = -4 + \log_5 3$ $x = -3.318$	$3^{2x} + 3^x - 12 = 0$ $(3^x)^2 + 3^x - 12 = 0$ Let $u = 3^x$ $u^2 + u - 12 = 0$ $(u-3)(u-4) = 0$ $u = 3,\ u = 4$ $3^x = 3,\ 3^x \neq 4$ $x = 1$

Logarithm Properties

Let b, m, and n be positive numbers such that $b \neq 1$.

Product Property $\log_b mn = \log_b m + \log_b n$

Quotient Property $\log_b \frac{m}{n} = \log_b m - \log_b n$

Power Property $\log_b m^n = n \log_b m$

Expand $\log_6 \frac{5x^3}{y}$.

$\log_6 \frac{5x^3}{y} = \log_6 5x^3 - \log_6 y$ **Quotient property**

$= \log_6 5 + \log_6 x^3 - \log_6 y$ **Product property**

$= \log_6 5 + 3\log_6 x - \log_6 y$ **Power property**

Change-of-Base Formula

If a, b, and c are positive numbers with $b \neq 1$ and $c \neq 1$, then:

$$\log_c a = \frac{\log_b a}{\log_b c}$$

In particular, $\log_c a = \frac{\log a}{\log c}$ and $\log_c a = \frac{\ln a}{\ln c}$.

Using common logarithms: $\log_3 8 = \frac{\log 8}{\log 3} \approx \frac{0.9031}{0.4771} \approx 1.893$

Using natural logarithms: $\log_3 8 = \frac{\ln 8}{\ln 3} \approx \frac{2.0794}{1.0986} \approx 1.893$

Solving Logarithmic Equations

**Goal – ONE term on each side*
**CHECK FOR EXTRANEOUS SOLUTIONS*

$3\ln(x-4)+2=-4$ $\ln(x-4)=-2$ $e^{\ln(x-4)}=e^{-2}$ $x-4=e^{-2}$ $x=4+e^{-2}$ $x=4.135$	$\log 2x+\log(x-5)=\log(100)$ $\log(2x^2-10x)=\log(100)$ $10^{\log(2x^2-10x)}=10^{\log(100)}$ $2x^2-10x=100$ $2x^2-10x-100=0$ $x^2-5x-50=0$ $(x-10)(x+5)=0$ $x=10, \quad x=-5$ $x=-5$ is extraneous, the answer is $x=10$

Compound Interest

Interest Compounded n times per year	Interest Compounded Continuously
$A=P\left(1+\frac{r}{n}\right)^{nt}$ • A is the final amount • P is the initial amount • r is the interest rate, in ***decimal*** form • t is the time in ***years*** • n is the number of times the interest is compounded per year	$A=Pe^{rt}$ • A is the final amount • P is the initial amount • r is the interest rate, in ***decimal*** form • t is the time in ***years***

Domain Restrictions

Fractions:

$y=\frac{f(x)}{g(x)}$	Domain: $\{x \mid g(x)\neq 0\}$	Ex: $y=\frac{x^2+9}{x-2}$, $\{x \mid x\neq 2\}$ Ex: $y=\frac{x-2}{x^2+9}$, $\{x \mid x\in R\}$

Even Roots:

$y=\sqrt[n]{f(x)}$, n is even	Domain: $\{x \mid f(x)\geq 0\}$	Ex: $y=2\sqrt{x+7}-3$, $\{x \mid x\geq -7\}$

Logarithms:

$y=\log_b(f(x))$	Domain: $\{x \mid f(x)>0\}$	Ex: $y=-2\ln x+4$, $\{x \mid x>0\}$